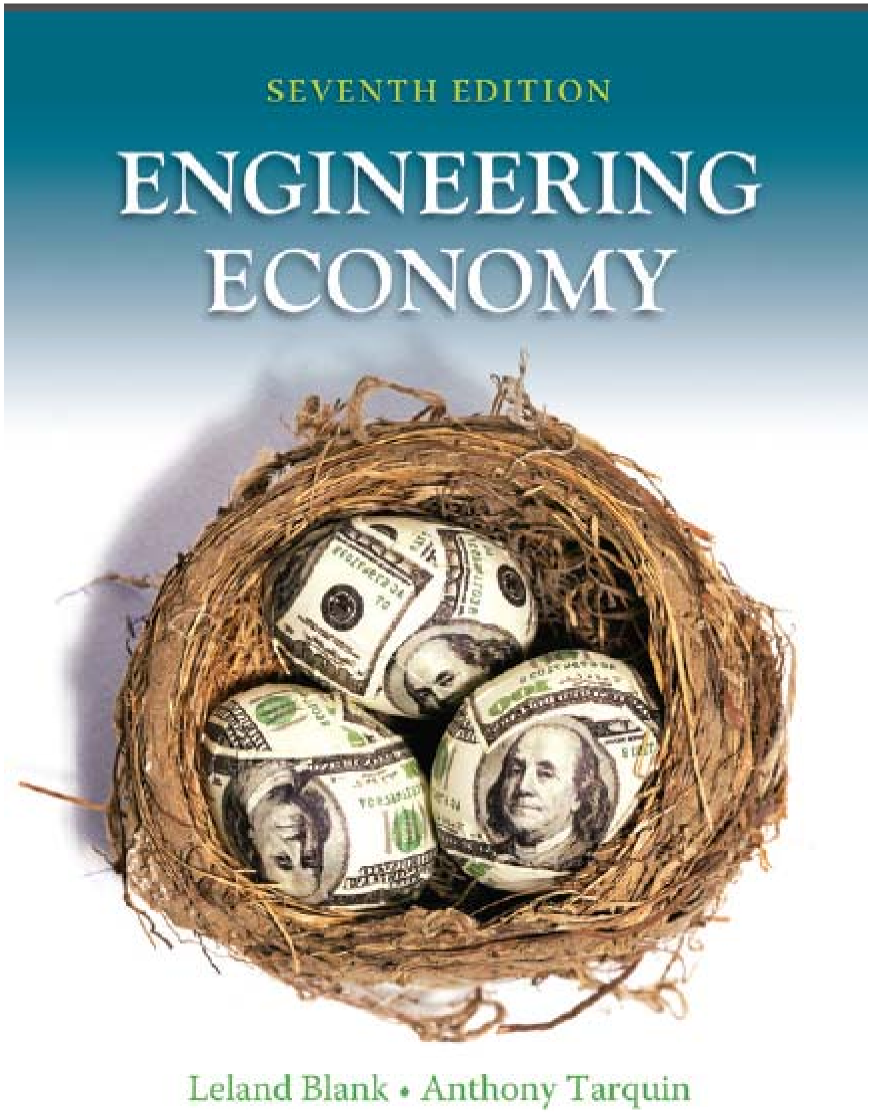


Chapter 2

Factors: How Time and Interest Affect Money

Lecture slides to accompany

Engineering Economy

7th edition

Leland Blank

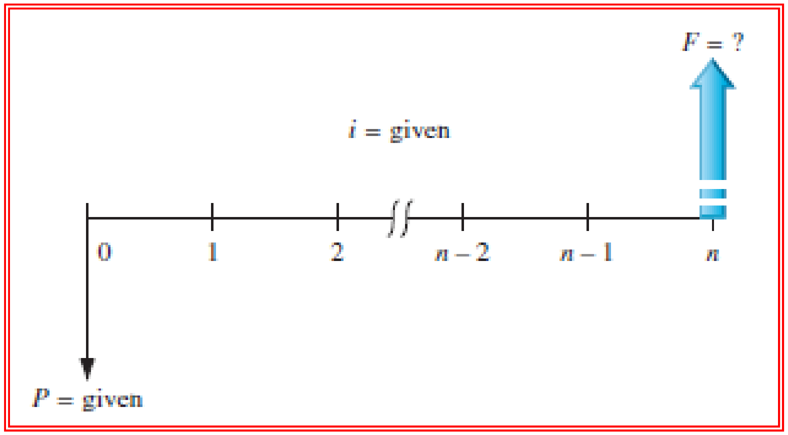
Anthony Tarquin

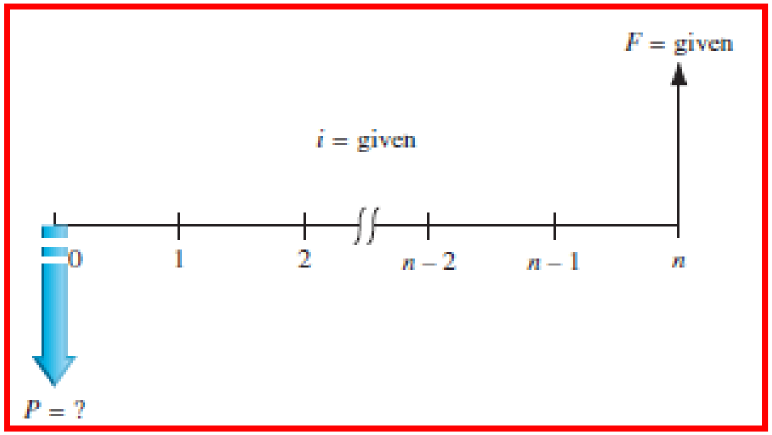
LEARNING OUTCOMES

1. **F/P and P/F Factors**
2. **P/A and A/P Factors**
3. **F/A and A/F Factors**
4. **Factor Values**
5. **Arithmetic Gradient**
6. **Geometric Gradient**
7. **Find i or n**

Single Payment Factors (F/P and P/F)

Single payment factors involve only P and F. Cash flow diagrams are as follows:





Formulas are as follows:

$F = P(1 + i)^n$

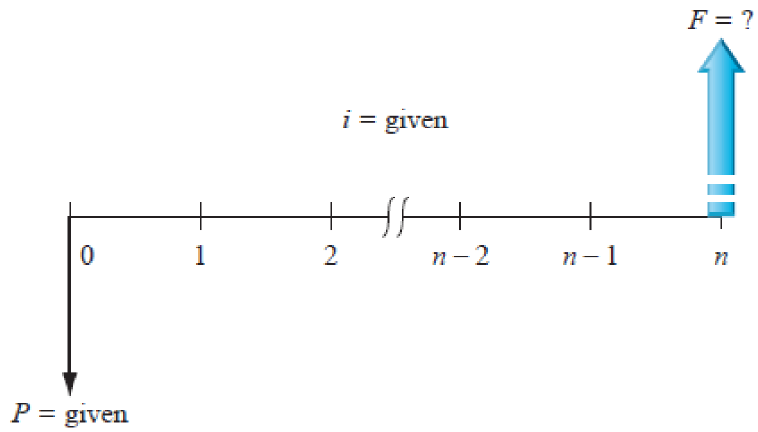
$P = F[1 / (1 + i)^n]$

Terms in parentheses or brackets are called *factors.* Values are in tables for i and n values

Factors are represented in *standard factor notation* *such as (F/P,i,n)*,

where letter to left of slash is what is sought; letter to right represents what is given

Single Payment Factors (F/P and P/F)



$$F_1 = P + Pi$$
$$= P(1 + i)$$
$$F_2 = F_1 + F_1 i$$
$$= P(1 + i) + P(1 + i)i$$

$$F_2 = P(1 + i + i + i^2)$$
$$= P(1 + 2i + i^2)$$
$$= P(1 + i)^2$$

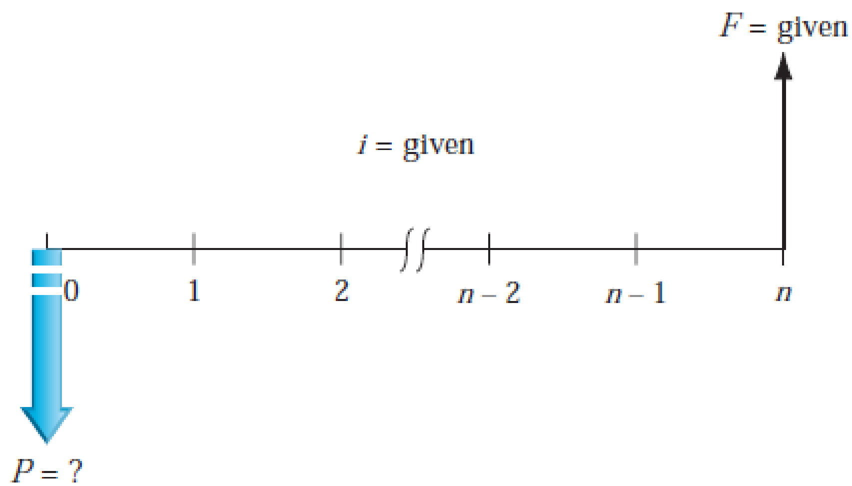
$$F_3 = F_2 + F_2 i$$

$$F_3 = P(1 + i)^3$$

$$F = P(1 + i)^n$$

The factor $(1 + i)^n$ is called the *single-payment compound amount factor* (SPCAF).

Single Payment Factors (F/P and P/F)

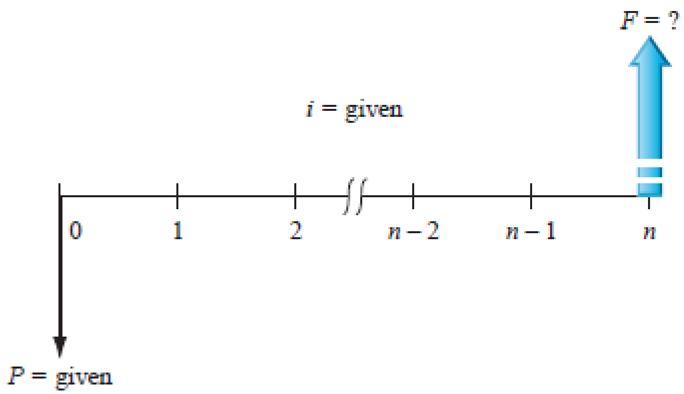


$$F = P(1 + i)^n$$

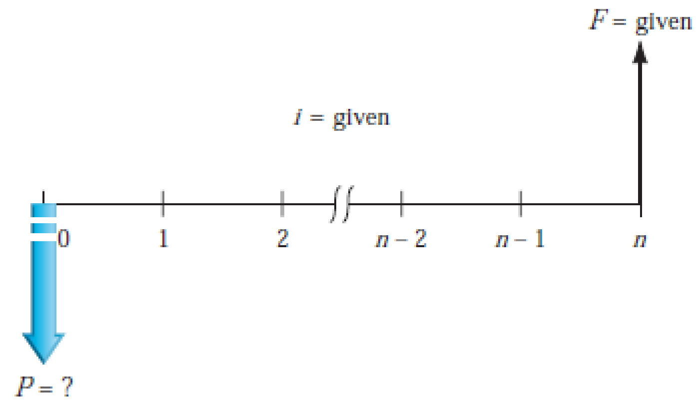
Reverse the situation to **determine the *P* value for a stated amount *F***

$$P = F\left[\frac{1}{(1 + i)^n}\right] = F(1 + i)^{-n}$$

The expression $(1 + i)^{-n}$ is known as the *single-payment present worth factor* (SPPWF)



(a)



(b)

TABLE 2–1 *F/P* and *P/F* Factors: Notation and Equations

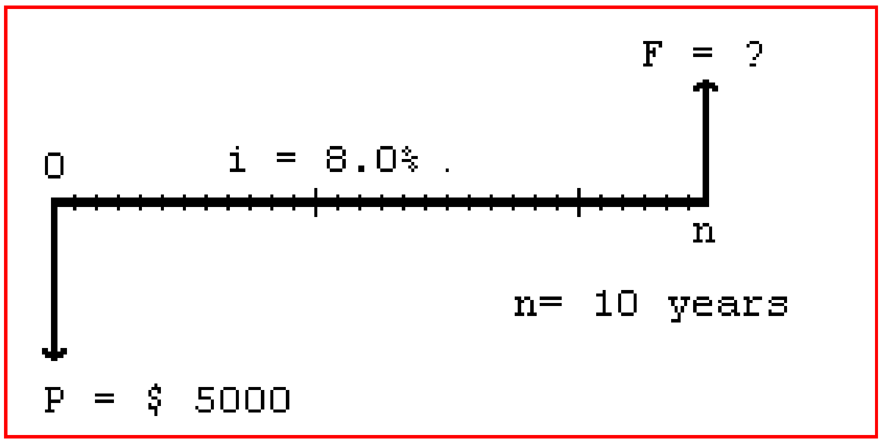
Factor Notation	Factor Name	Find/Given	Standard Notation Equation	Equation with Factor Formula	Excel Function
$(F/P,i,n)$	Single-payment compound amount	F/P	$F = P(F/P,i,n)$	$F = P(1 + i)^n$	$= FV(i\%,n,,P)$
$(P/F,i,n)$	Single-payment present worth	P/F	$P = F(P/F,i,n)$	$P = F(1 + i)^{-n}$	$= PV(i\%,n,,F)$

Example: Finding Future Value

A person deposits $5000 into an account which pays interest at a rate of 8% per year. The amount in the account after 10 years is closest to:

(A) $2,792 (B) $9,000 (C) $10,795 (D) $12,165

The cash flow diagram is:



Solution:

$F = P(F/P,i,n)$

$= 5000(F/P,8\%,10)$

$= 5000(2.1589)$

$= \$10,794.50$

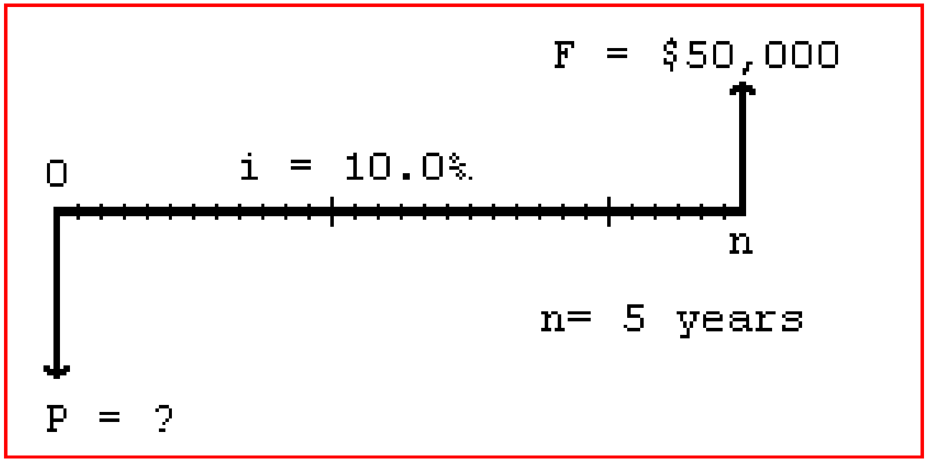
Answer is (C)

Example: Finding Present Value

A small company wants to make a single deposit now so it will have enough money to purchase a backhoe costing $50,000 five years from now. If the account will earn interest of 10% per year, the amount that must be deposited now is nearest to:

(A) $10,000   (B) $ 31,050   (C) $ 33,250   (D) $319,160

The cash flow diagram is:



Solution:

P = F(P/F,i,n)

= 50,000(P/F,10%,5)

= 50,000(0.6209)

= $31,045

Answer is (B)

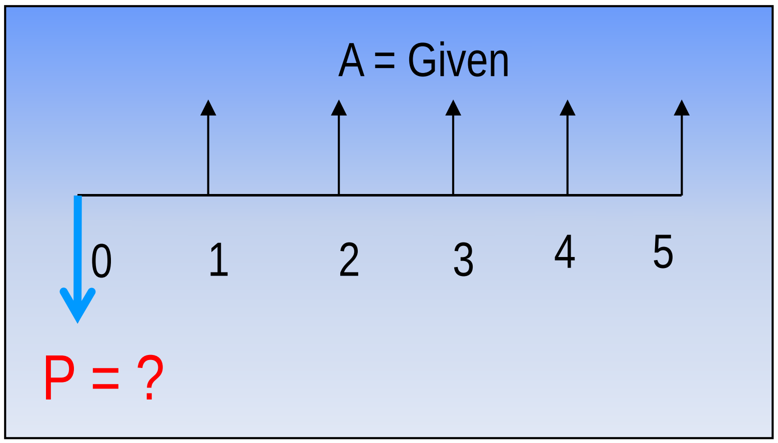
Uniform Series Involving P/A and A/P

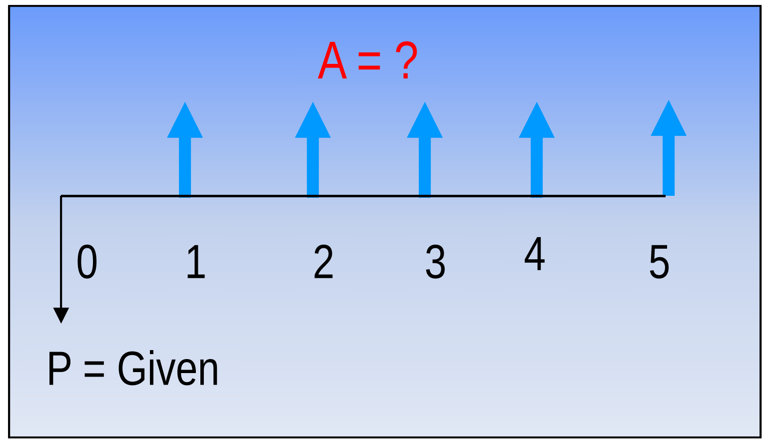
The uniform series factors that involve P and A are derived as follows:

(1) Cash flow occurs in *consecutive* interest periods

(2) Cash flow amount is *same* in each interest period

The cash flow diagrams are:

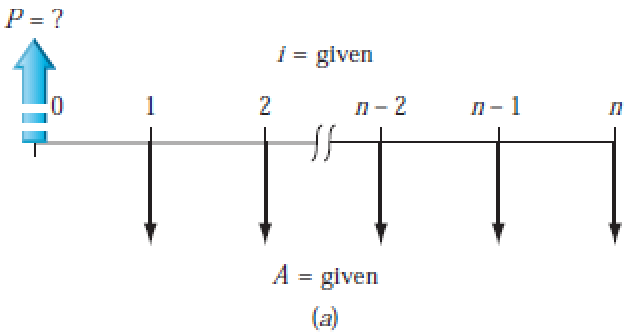




P = A(P/A,i,n) ⟵ Standard Factor Notation ⟶ A = P(A/P,i,n)

Note: P is one period *Ahead* of first A value

Uniform Series Involving P/A and A/P



$$P = A\left[\frac{1}{(1+i)^1}\right] + A\left[\frac{1}{(1+i)^2}\right] + A\left[\frac{1}{(1+i)^3}\right] + \cdots + A\left[\frac{1}{(1+i)^{n-1}}\right] + A\left[\frac{1}{(1+i)^n}\right]$$

$$P = A\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right]$$

Term inside the brackets is a geometric progression. Multiply the equation by 1/(1+i) to yield a second equation

$$\frac{P}{1+i} = A\left[\frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \ldots + \frac{1}{(1+i)^n} + \frac{1}{(1+i)^{n+1}}\right]$$

$$\frac{1}{1+i}P = A\left[\frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \ldots + \frac{1}{(1+i)^n} + \frac{1}{(1+i)^{n+1}}\right]$$

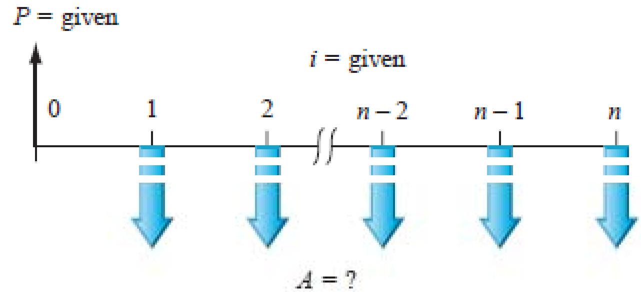
$$-\quad P = A\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right]$$

$$\frac{-i}{1+i}P = A\left[\frac{1}{(1+i)^{n+1}} - \frac{1}{(1+i)^1}\right]$$

$$P = \frac{A}{-i}\left[\frac{1}{(1+i)^n} - 1\right]$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \qquad i \neq 0$$

Uniform Series Involving P/A and A/P



(b)

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \qquad i \neq 0$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

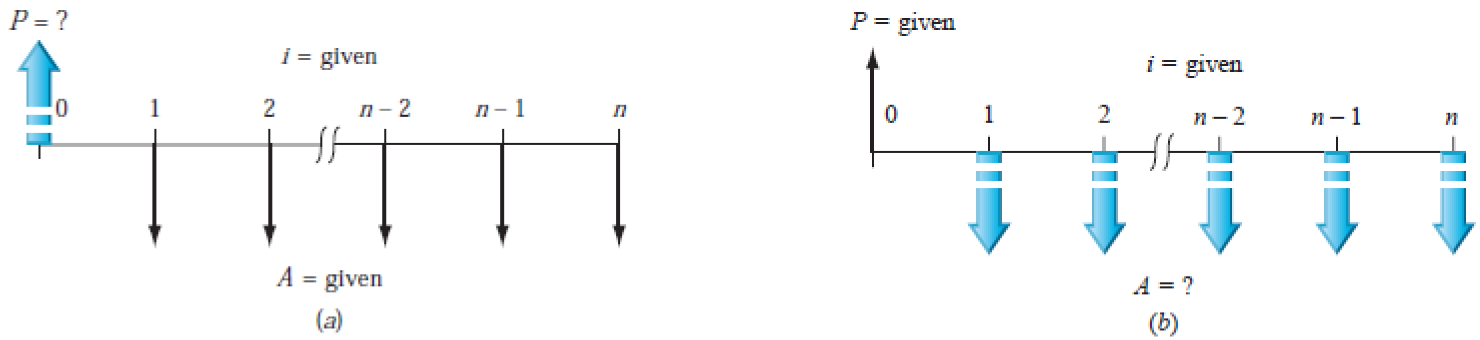


TABLE 2–2 *P/A* and *A/P* Factors: Notation and Equations

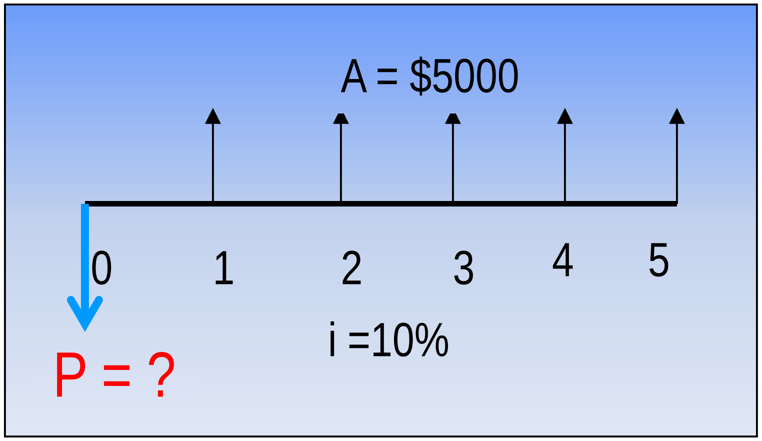
Factor Notation	Factor Name	Find/Given	Factor Formula	Standard Notation Equation	Excel Function
$(P/A,i,n)$	Uniform series present worth	P/A	$\frac{(1+i)^n - 1}{i(1+i)^n}$	$P = A(P/A,i,n)$	= PV(i%,n,A)
$(A/P,i,n)$	Capital recovery	A/P	$\frac{i(1+i)^n}{(1+i)^n - 1}$	$A = P(A/P,i,n)$	= PMT(i%,n,P)

Example: Uniform Series Involving P/A

A chemical engineer believes that by modifying the structure of a certain water treatment polymer, his company would earn an extra $5000 per year. At an interest rate of 10% per year, how much could the company afford to spend now to just break even over a 5 year project period?

(A) $11,170 (B) 13,640 (C) $15,300 (D) $18,950

The cash flow diagram is as follows:



Solution:

P = 5000(P/A,10%,5)

= 5000(3.7908)

= $18,954

Answer is (D)

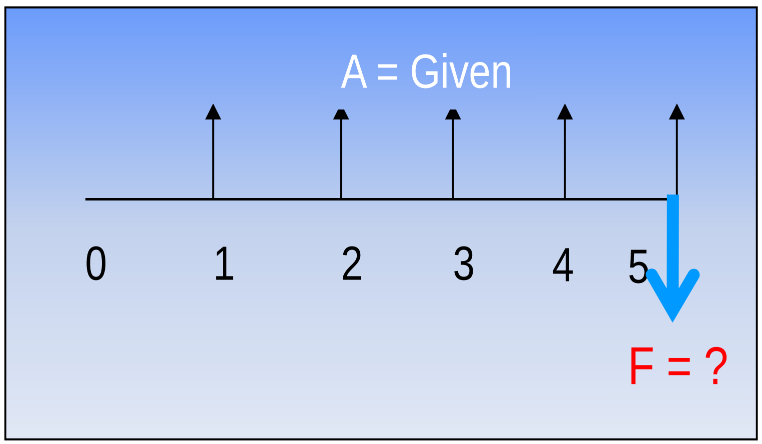
Uniform Series Involving F/A and A/F

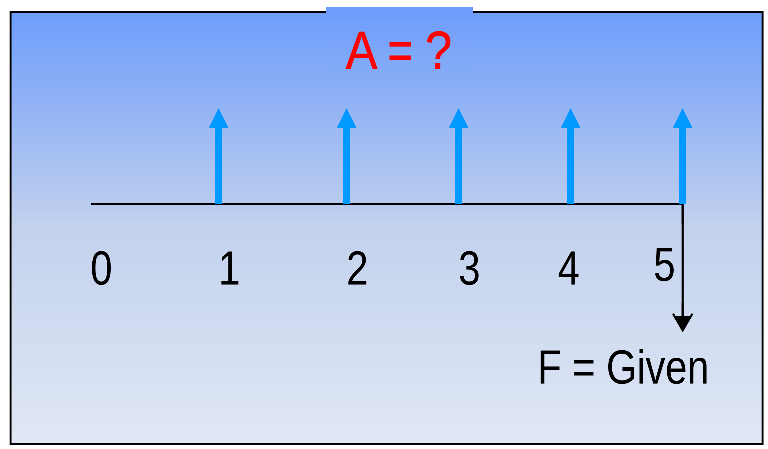
The uniform series factors that involve F and A are derived as follows:

(1) Cash flow occurs in *consecutive* interest periods

(2) Last cash flow occurs in *same* period as F

Cash flow diagrams are:





F = A(F/A,i,n) ⟸ Standard Factor Notation ⟹ A = F(A/F,i,n)

Note: F takes place in the *same* period as last A

Uniform Series Involving F/A and A/F

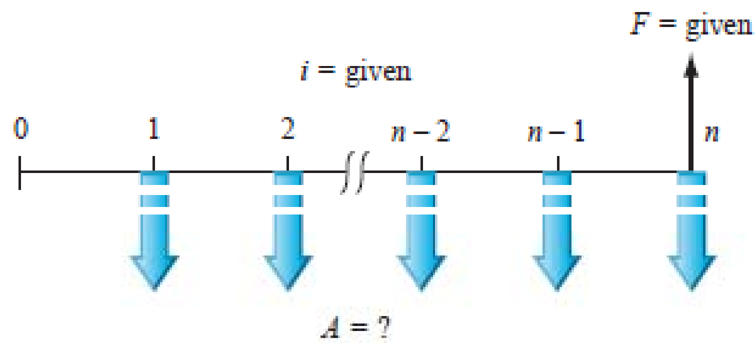
- Take advantage of what we already have
- Recall:

$$P = F\left[\frac{1}{(1+i)^n}\right]$$

- Also:

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

Substitute "P" and simplify!



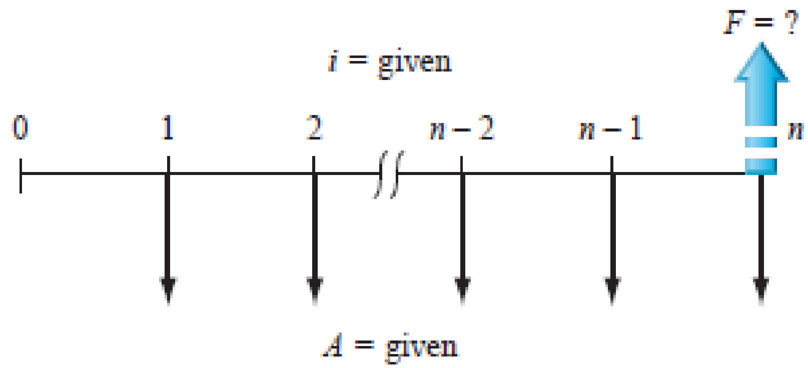
- By substitution we see:

$$A = F\left[\frac{1}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

- Simplifying we have:
- Which is the (A/F,i%,n) factor

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

Uniform Series Involving F/A and A/F



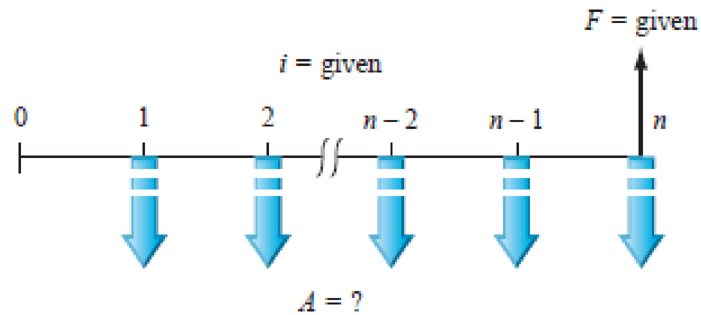
- Given:

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

- Solve for F in terms of A

$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

Uniform Series Involving F/A and A/F



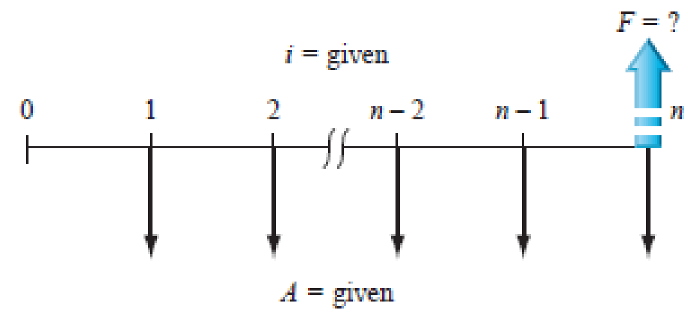


TABLE 2–3 ***F/A* and *A/F* Factors: Notation and Equations**

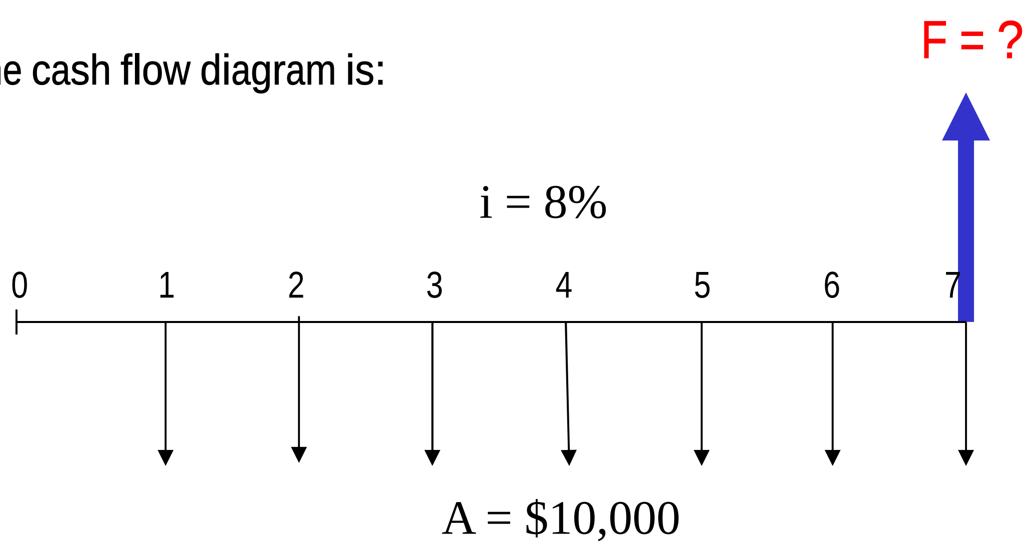
Factor Notation	Factor Name	Find/Given	Factor Formula	Standard Notation Equation	Excel Functions
$(F/A,i,n)$	Uniform series compound amount	F/A	$\frac{(1+i)^n - 1}{i}$	$F = A(F/A,i,n)$	= FV(i%,n,A)
$(A/F,i,n)$	Sinking fund	A/F	$\frac{i}{(1+i)^n - 1}$	$A = F(A/F,i,n)$	= PMT(i%,n,F)

Example: Uniform Series Involving F/A

An industrial engineer made a modification to a chip manufacturing process that will save her company $10,000 per year. At an interest rate of 8% per year, how much will the savings amount to in 7 years?

(A) $45,300 (B) $68,500 (C) $89,228 (D) $151,500

The cash flow diagram is:



Solution:

$$F = 10{,}000(F/A,8\%,7)$$

$$= 10{,}000(8.9228)$$

$$= \$89{,}228$$

Answer is (C)

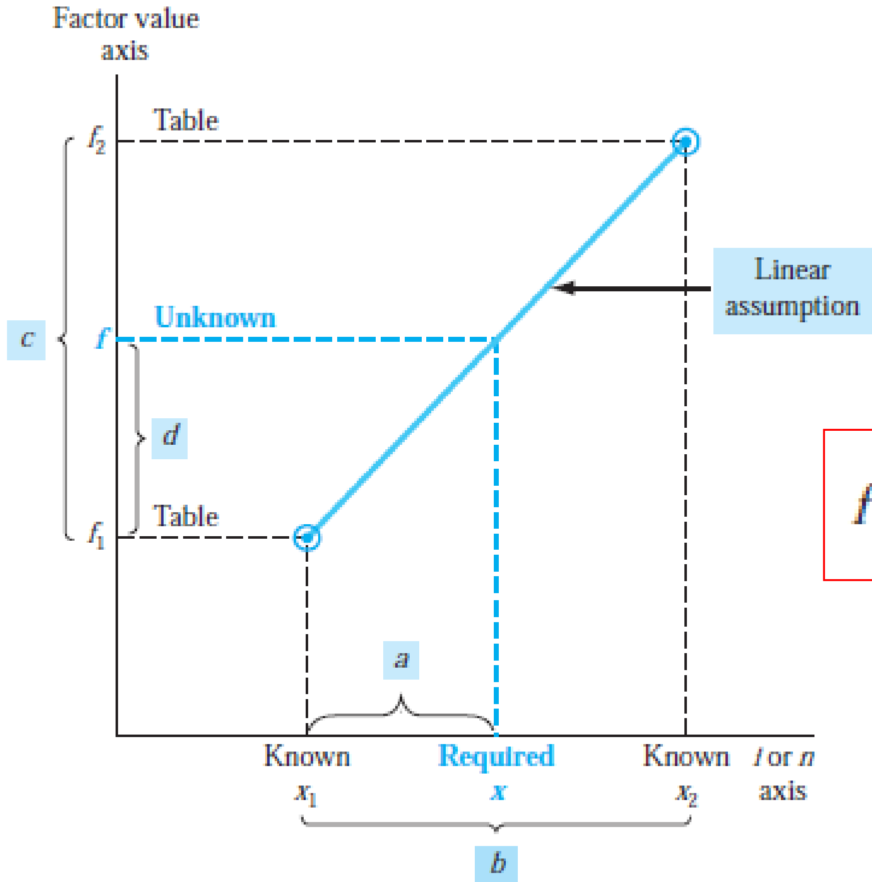
Factor Values for Untabulated i or n

3 ways to find factor values for untabulated i or n values

- Use formula
- Use spreadsheet function with corresponding P, F, or A value set to 1
- Linearly interpolate in interest tables

Formula or spreadsheet function is fast and accurate
Interpolation is only approximate

Factor Values for Untabulated i or n



$$f = f_1 + \frac{(x - x_1)}{(x_2 - x_1)}(f_2 - f_1)$$

Example: Untabulated i

Determine the value for (F/P, 8.3%,10)

Formula: $F = (1 + 0.083)^{10} = 2.2197$ ⟸ OK

Spreadsheet: = FV(8.3%,10,,1) = 2.2197 ⟸ OK

Interpolation: 8% ------ 2.1589
8.3% ------ x
9% ------ 2.3674

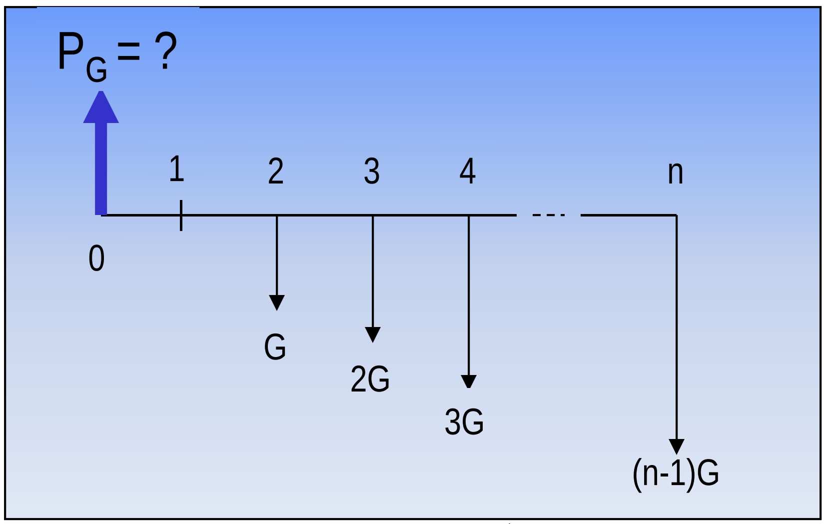
x = 2.1589 + [(8.3 - 8.0)/(9.0 - 8.0)][2.3674 – 2.1589]
= 2.2215 ⟸ (Too high)

Absolute Error = 2.2215 – 2.2197 = 0.0018

Arithmetic Gradients

Arithmetic gradients *change* by the *same amount* each period

The cash flow diagram for the P_G of an arithmetic gradient is:



Standard factor notation is

$P_G = G(P/G,i,n)$

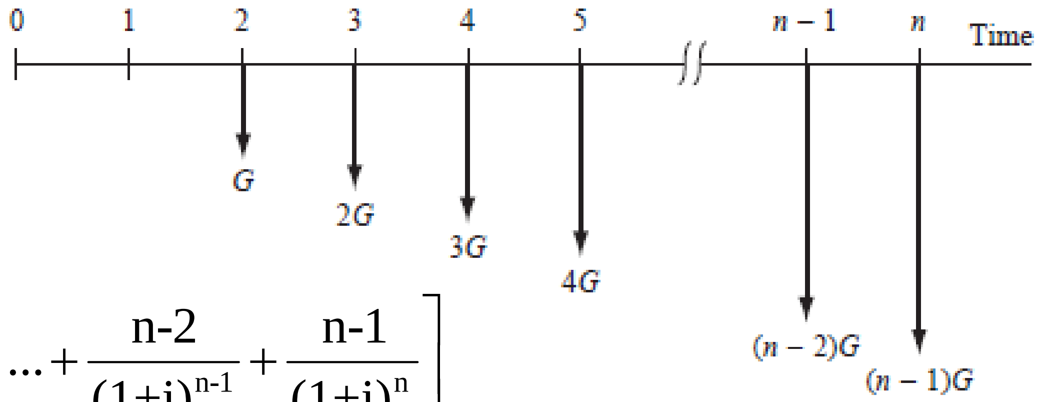
G starts between periods 1 and 2
(not between 0 and 1)

This is because cash flow in year 1 is usually not equal to G and is handled separately as a *base amount*

Note that P_G is located Two Periods Ahead of the first change that is equal to G

Remember: The conventional arithmetic gradient starts in year 2, and *P* is located in year 0.

Arithmetic Gradients



$$P=G\left[\frac{1}{(1+i)^2}+\frac{2}{(1+i)^3}+\ldots+\frac{n-2}{(1+i)^{n-1}}+\frac{n-1}{(1+i)^n}\right]$$

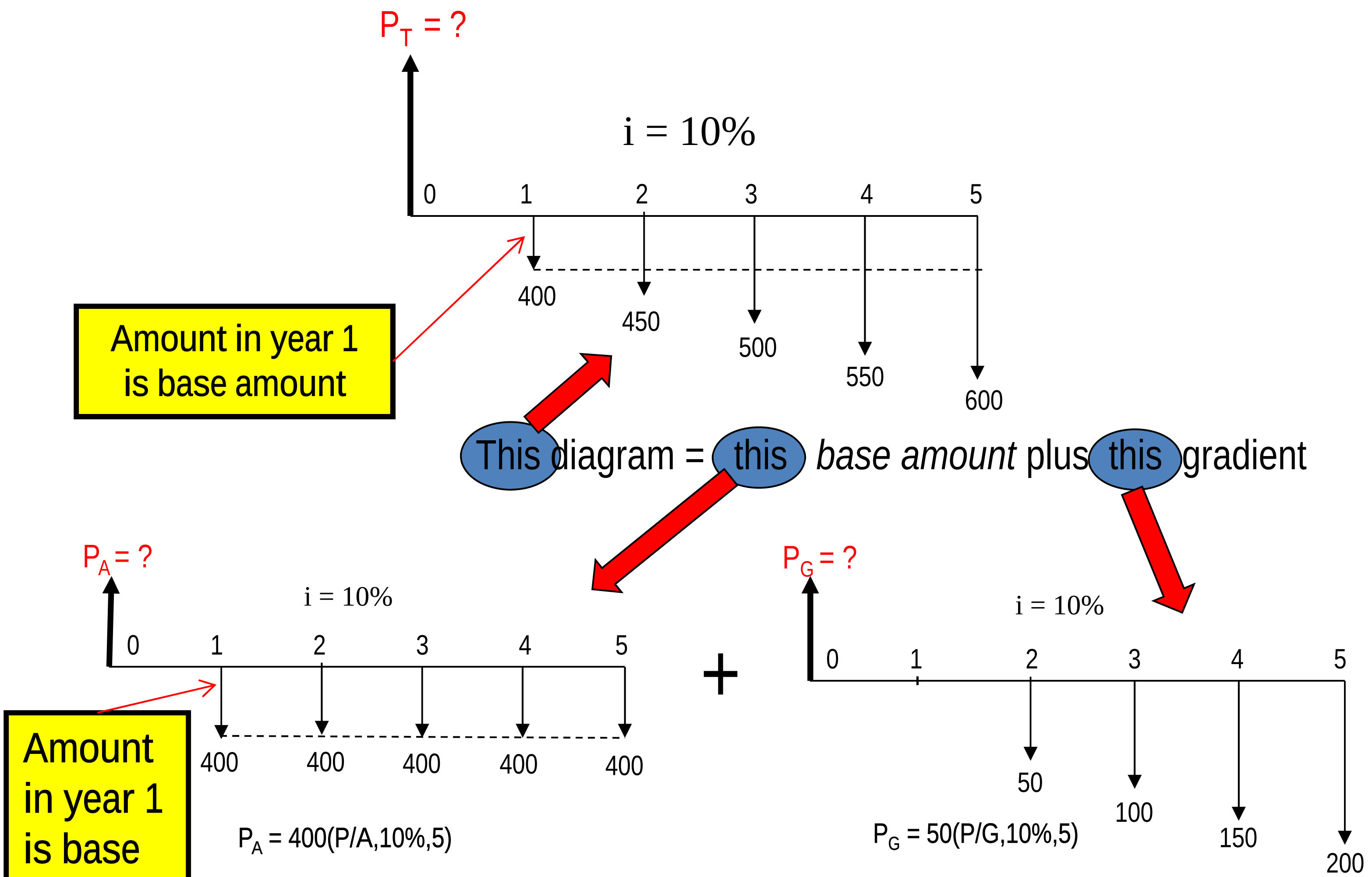
Multiply both sides by (1+i)

$$P(1+i)^1=G\left[\frac{1}{(1+i)^1}+\frac{2}{(1+i)^2}+\ldots+\frac{n-2}{(1+i)^{n-2}}+\frac{n-1}{(1+i)^{n-1}}\right]$$

Subtracting [1] from [2].....

$$P_G=\frac{G}{i}\left[\frac{(1+i)^n-1}{i(1+i)^n}-\frac{n}{(1+i)^n}\right]$$

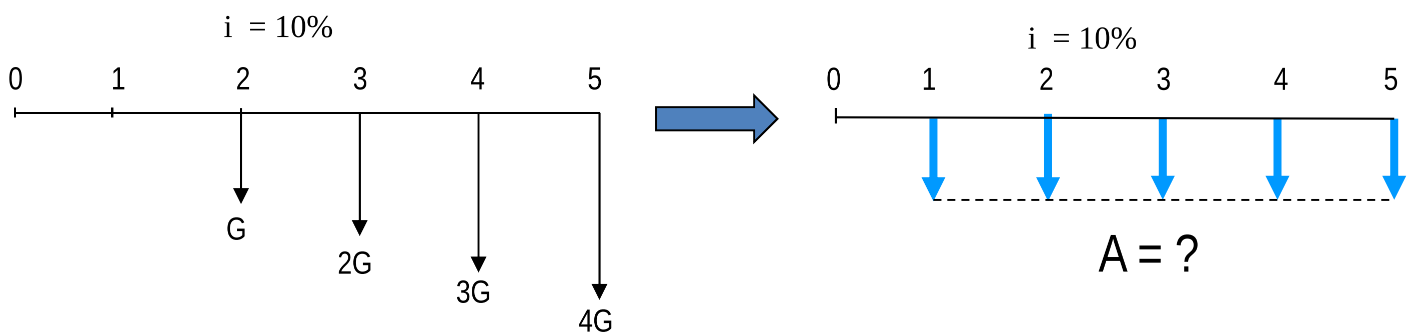
Typical Arithmetic Gradient Cash Flow



$$P_T = P_A + P_G = 400(P/A,10\%,5) + 50(P/G,10\%,5)$$

Converting Arithmetic Gradient to A

Arithmetic gradient can be converted into equivalent A value using G(A/G,i,n)



$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

$$P_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$$

$$A_G = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

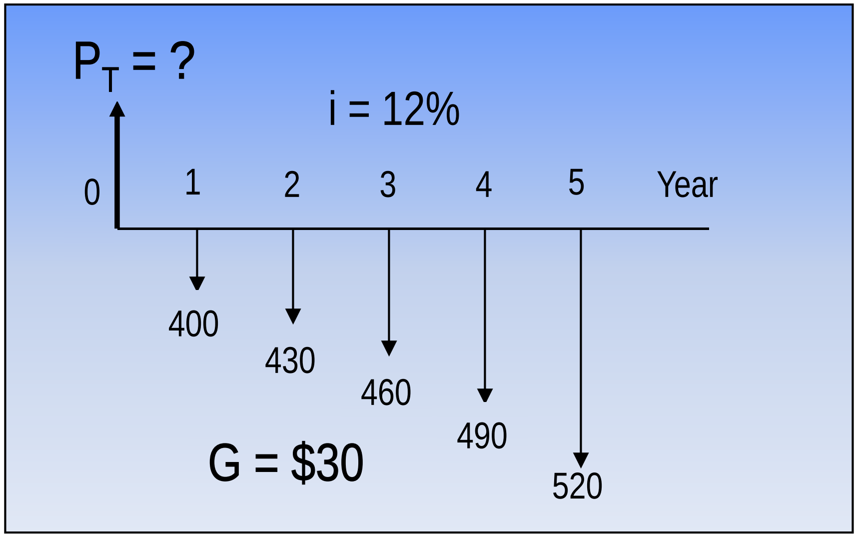
$$A_T = A_A \pm A_G$$

General equation when base amount is involved is A = base amount + G(A/G,i,n)

Example: Arithmetic Gradient

The present worth of \$400 in year 1 and amounts increasing by \$30 per year through year 5 at an interest rate of 12% per year is closest to:

(A) \$1532 (B) \$1,634 (C) \$1,744 (D) \$1,829



Solution:

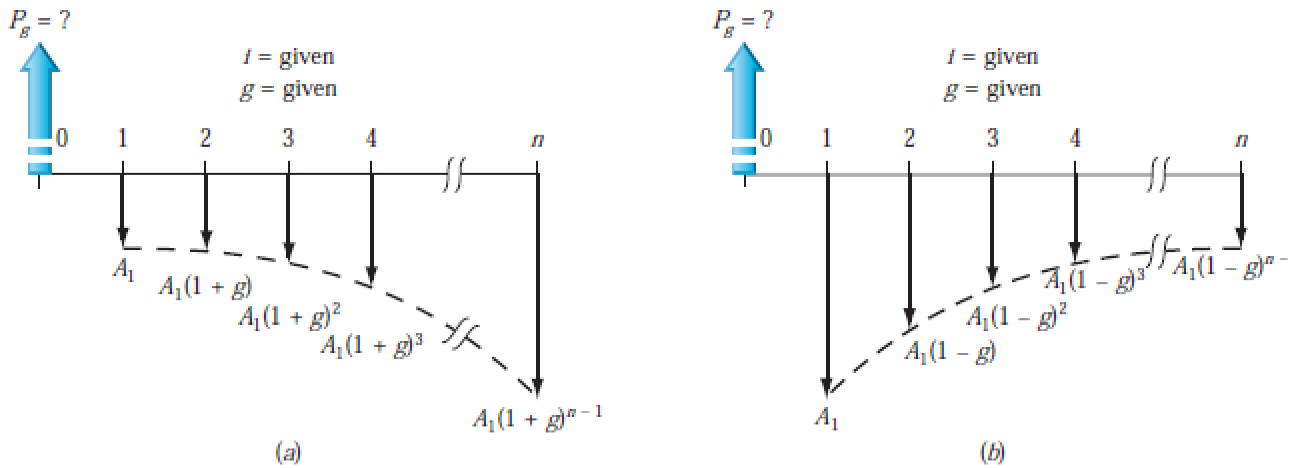
$P_T = 400(P/A,12\%,5) + 30(P/G,12\%,5)$
$= 400(3.6048) + 30(6.3970)$
$= \$1{,}633.83$

Answer is (B)

The cash flow could also be converted into an A value as follows:

$A = 400 + 30(A/G,12\%,5)$
$= 400 + 30(1.7746)$
$= \$453.24$

Geometric Gradients



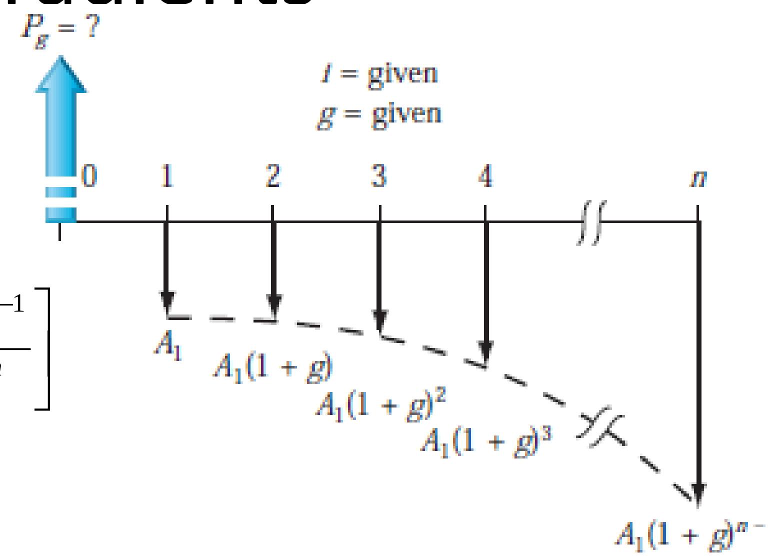
A **geometric gradient** series is a cash flow series that either increases or decreases by a **constant percentage** each period. The uniform change is called the **rate of change.**

g = **constant rate of change**, in decimal form, by which cash flow values increase or decrease from one period to the next. The gradient g can be + or −.

A_1 = **initial cash flow in year 1** of the geometric series

P_g = **present worth** of the entire geometric gradient series, including the initial amount A_1

Geometric Gradients



$$P_g = A_1\left[\frac{1}{(1+i)} + \frac{(1+g)^1}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \ldots + \frac{(1+g)^{n-1}}{(1+i)^n}\right]$$

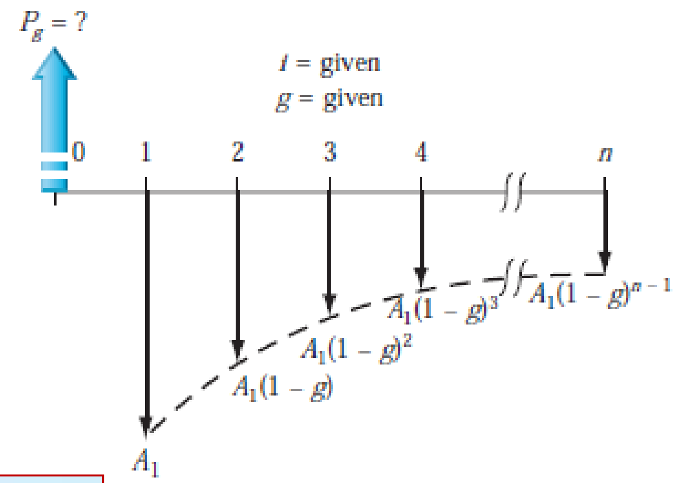
Multuply both sides by $\frac{(1+g)}{(1+i)}$ to create another equat

$$P_g\frac{(1+g)}{(1+i)} = A_1\frac{(1+g)}{(1+i)}\left[\frac{1}{(1+i)} + \frac{(1+g)^1}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \ldots + \frac{(1+g)^{n-1}}{(1+i)^n}\right]$$

$$P_g\left(\frac{1+g}{1+i} - 1\right) = A_1\left[\frac{(1+g)^n}{(1+i)^{n+1}} - \frac{1}{1+i}\right]$$

$$P_g = A_1\left[\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}\right] \quad g \neq i$$

Geometric Gradients



<u>For the case i = g</u>

$$P_g = A_1\left(\frac{1}{(1+i)} + \frac{1}{(1+i)} + \frac{1}{(1+i)} + \ldots + \frac{1}{(1+i)}\right)$$

$$P_g = \frac{nA_1}{(1+i)}$$

$$P_g = A_1(P/A,g,i,n)$$

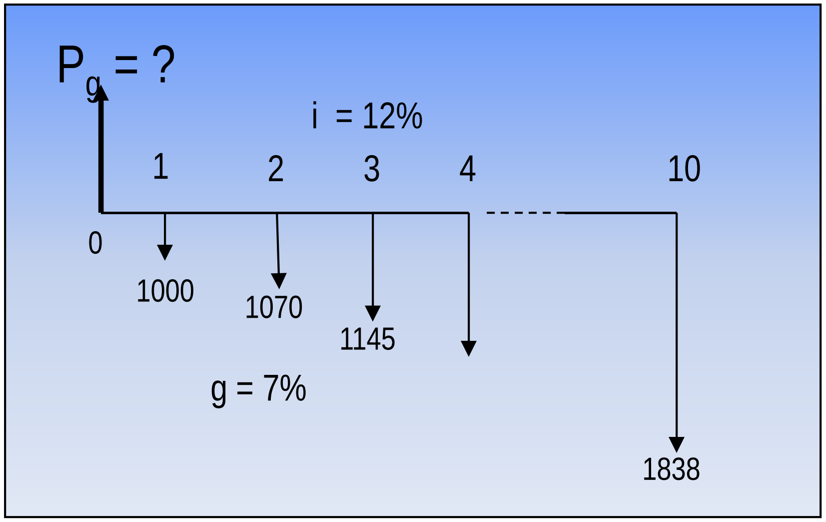
$$(P/A,g,i,n) = \begin{cases} \dfrac{1-\left(\dfrac{1+g}{1+i}\right)^n}{i-g} & g \neq i \\ \dfrac{n}{1+i} & g = i \end{cases}$$

The $(P/A,g,i,n)$ factor calculates P_g in period $t = 0$ for a geometric gradient series **starting in period 1** in the amount A_1 and increasing by a constant rate of g each period.

Example: Geometric Gradient

Find the present worth of $1,000 in year 1 and amounts increasing by 7% per year through year 10. Use an interest rate of 12% per year.

(a) $5,670 (b) $7,333 (c) $12,670 (d) $13,550



Solution:

P_g = 1000[1-(1+0.07/1+0.12)^{10}]/(0.12-0.07)

= $7,333

Answer is (b)

To find A, multiply P_g by (A/P,12%,10)

Unknown Interest Rate i

Unknown interest rate problems involve solving for i,
given n and 2 other values (P, F, or A)

(Usually requires a trial and error solution or interpolation in interest tables)

Procedure: Set up equation with all symbols involved and solve for i

A contractor purchased equipment for $60,000 which provided income of $16,000 per year for 10 years. The annual rate of return of the investment was closest to:

(a) 15% (b) 18% (c) 20% (d) 23%

Solution: Can use either the P/A or A/P factor. Using A/P:

$$60{,}000(A/P,i\%,10) = 16{,}000$$

$$(A/P,i\%,10) = 0.26667$$

From A/P column at n = 10 in the interest tables, i is between 22% and 24% Answer is (d)

Unknown Recovery Period n

Unknown recovery period problems involve solving for n, given i and 2 other values (P, F, or A)

(Like interest rate problems, they usually require a trial & error solution or interpolation in interest tables)

Procedure: Set up equation with all symbols involved and solve for n

A contractor purchased equipment for $60,000 that provided income of $8,000 per year. At an interest rate of 10% per year, the length of time required to recover the investment was closest to:

(a) 10 years (b) 12 years (c) 15 years (d) 18 years

Solution: Can use either the P/A or A/P factor. Using A/P:

$$60{,}000(A/P,10\%,n) = 8{,}000$$

$$(A/P,10\%,n) = 0.13333$$

From A/P column in i = 10% interest tables, n is between 14 and 15 years Answer is (c)

Summary of Important Points

- In P/A and A/P factors, P is *one period ahead* of first A
- In F/A and A/F factors, F is in *same period as last* A
- To find untabulated factor values, best way is to use *formula or spreadsheet*
- For arithmetic gradients, gradient G starts between *periods 1 and 2*
- Arithmetic gradients have 2 parts, *base amount* (year 1) and *gradient amount*
- For geometric gradients, gradient g starts been *periods 1 and 2*
- In geometric gradient formula, A_1 is amount in *period 1*
- To find unknown i or n, *set up equation involving all terms* and solve for i or n