Structural Mechanics (CE- 312)
Buckling and Stability of Columns

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DESIGN CONSIDERATION OF STRUCTURE

1: Strength: The ability of structure to support a specified load without experiencing excessive load.

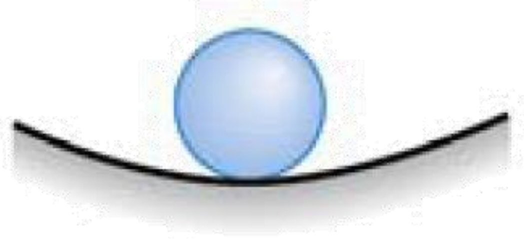
2: Deformation: The ability of structure to support a specified load without undergoing appreciable deformation.

3: Stability: The ability of structure or structural member to support a given load without experiencing a sudden change in its configuration (Buckling).

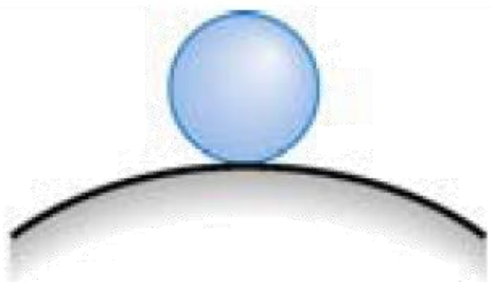
We define instability instead of stability

- Change in geometry of a structure or structural component under compression , resulting in loss of ability to resist loading is defined as *instability.*

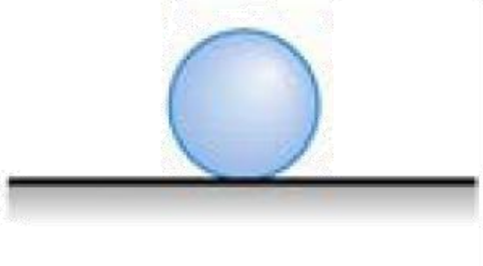
- Structure is in *unstable equilibrium* when small perturbations (disturbance) produce large movements and the structure never returns to its original equilibrium position.
- Structure is in *neutral equilibrium* when we cant decide whether it is in stable or unstable equilibrium. Small perturbation cause large movements but the structure can be brought back to its original equilibrium position with no work.
- Thus, stability talks about the equilibrium state of the structure.



Stable Equilibrium



Unstable Equilibrium



Neutral Equilibrium

- The definition of stability had nothing to do with a change in the geometry of the structure under compression.
- Change in geometry of structure under compression that results in its ability to resist loads called *instability*.
- Not true :this is called *buckling*.
- *Buckling* is a phenomenon that can occur for structures under compressive loads.

Stability of equilibrium:

- As the loads acting on the structure are increased, when does the equilibrium state become unstable?
- The equilibrium state becomes unstable due to:
  - *Large deformations of the structure*
  - *Inelasticity of the structural materials*

COLUMN

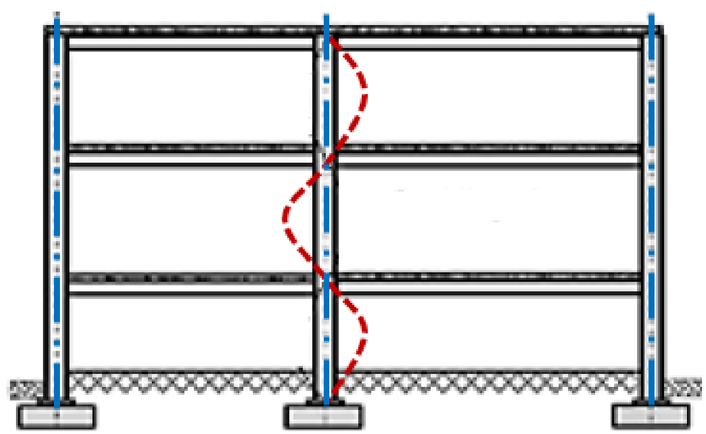
A column is a line element (long slender bar) subjected to axial compression. The term is frequently used to describe a vertical member.

- Structural members (*i.e.*, columns) are generally stable when subjected to tensile loading and fail when the stress in the cross section exceeds the ultimate strength of material.
- In case of elements (*i.e.*, column) subjected to the compressive loading, secondary bending effect *e.g.*, imperfections within material and/or fabrication process, inaccurate positioning of loads or asymmetry of cross section can induce premature failure either in part of cross section or of the whole element. In such case failure mode is normally the *Buckling*.

Buckling is categorized into the following

1. Overall buckling
2. Local buckling
3. Lateral Torsional buckling

- The design of the most of the compressive members is governed by over-all buckling capacity. ***i.e.,*** the maximum compressive load which can be carried before the failure occurs due to the excessive deflection in the plane of greatest slenderness ratio.
- Typical overall buckling occur in columns of frame structure and in compression members of trusses



SLENDERNESS RATIO (L_e/r_{min})

It is the ratio of the *effective length of column* (L_e) to the *minimum radius of gyration* (r_{min}) of cross sectional area.

- If the columns is free to rotate at each end then buckling takes place about that axes for which the radius of gyration is minimum.

TYPES OF THE COLUMNS

The compression elements (Columns) are sub-divided into the following three categories.

1. Short Column

The column which has a relatively low slenderness ratio is called the short column (***e.g.***, length of not greater than the 10 time to the least cross sectional dimension).

- Failure occur when stress over the cross section reaches the yield or crushing value of the material.

- Such element fail by crushing of material induced by predominantly axial compressive stress (flexure stresses are not dominant).

2. Slender Column

The column which has a relatively high slenderness ratio is called the slender or long column (***e.g.***, length is greater than the 30 time to the least cross sectional dimension).

- Such element fail due to excessive lateral deflection (*i.e.*, buckling) at a value of stress considerably less than the yield or crushing value.
- In slender column flexure stress are dominant and compressive stress are not too important.

3. Intermediate Column

The failure of columns is neither short nor slender and occur due the combination buckling and yielding/crushing.

- For Intermediate column Length is in between 10 to 30 time to the least cross sectional dimension.

Ideal Column

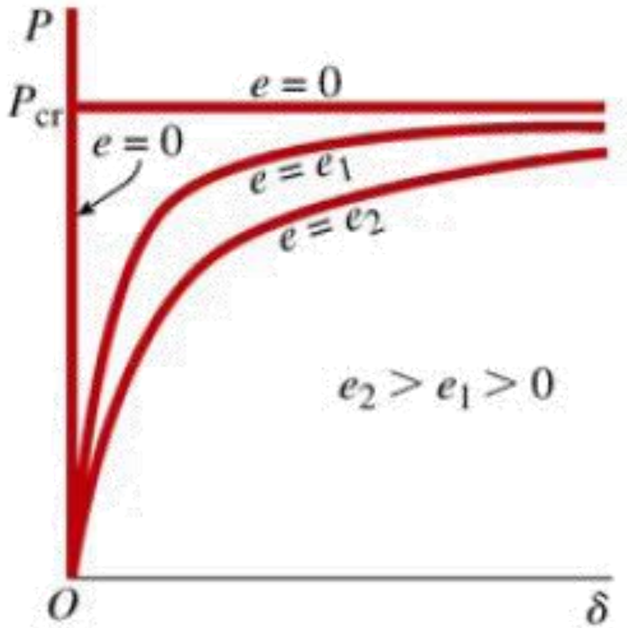
An ideal column has the following properties.

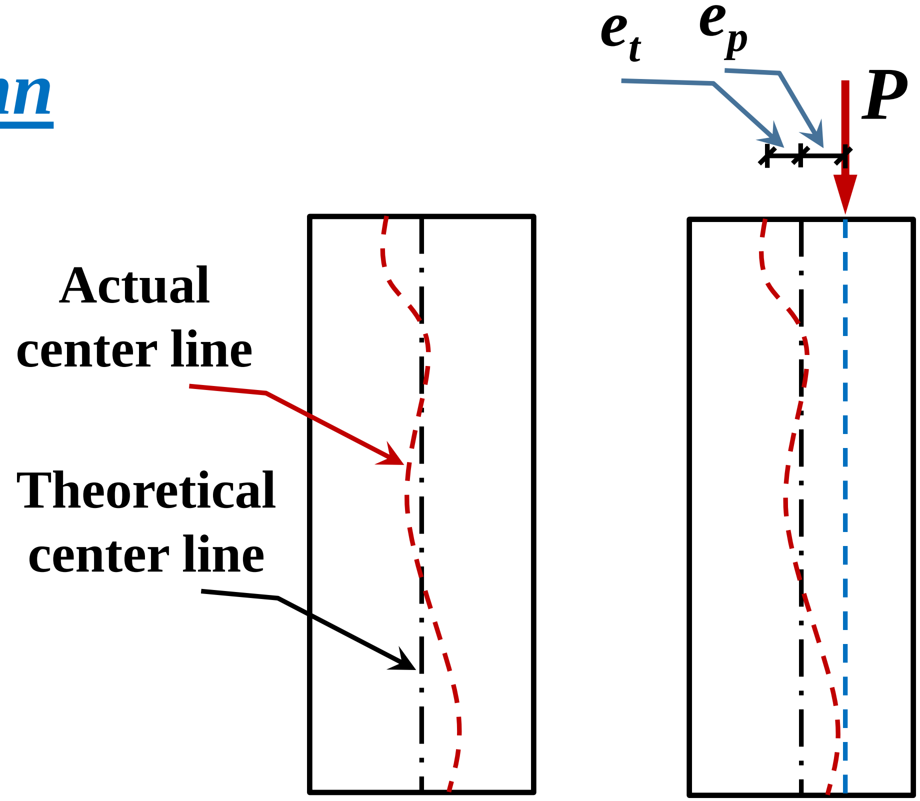
1. Its is prismatic (having the constant cross section through out the length).
2. Material is homogeneous.
3. Loading is perfectly axial.
4. Pin ended condition (simply supported) are frictionless.

Real Column

1. Imperfection are present (*i.e.*, structural and geometric)
2. Its not perfectly prismatic
3. Centroid may not lie on line joining the centroid of the end section.
4. Load is not acting along the centroidal line.

Stress in Eccentric Column





e = Total eccentricity
e_t = Theoretical eccentricity
e_p = Loading eccentricity
Z = section Modulus

$$e = e_t + e_p$$

$$\sigma = \frac{P}{A} \pm \frac{P.e}{Z}$$

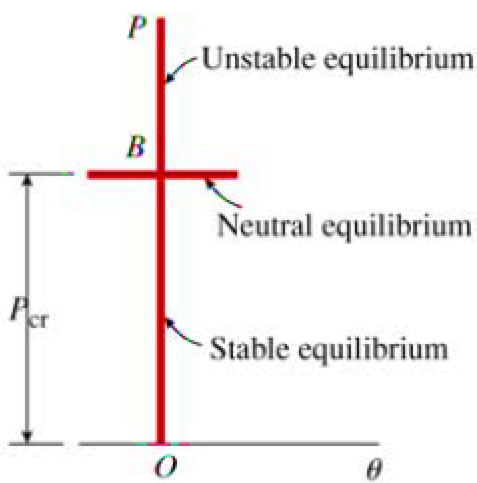
CRITICAL LOAD OF COLUMNS

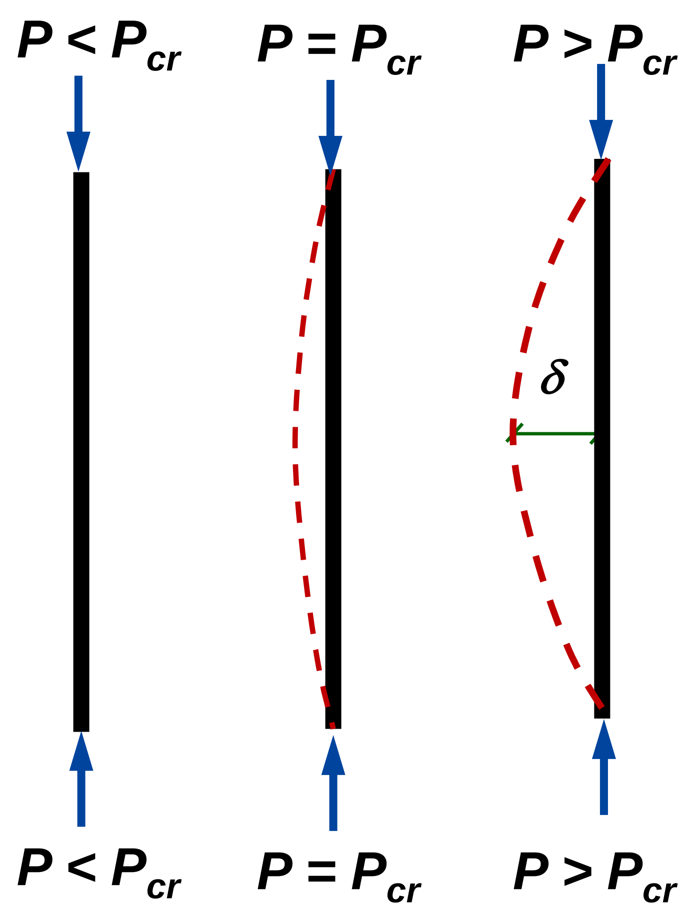
The critical load of as slender bar (columns) subjected to axial compression is that value of the axial load that is just sufficient to keep the bar a slightly deflected configuration.

Case-I: $P < P_{cr}$
Stable Equilibrium and No Buckling

Case-II: $P = P_{cr}$
Equilibrium State and Slight deflection

Case-III: $P > P_{cr}$
Unstable State and Buckling

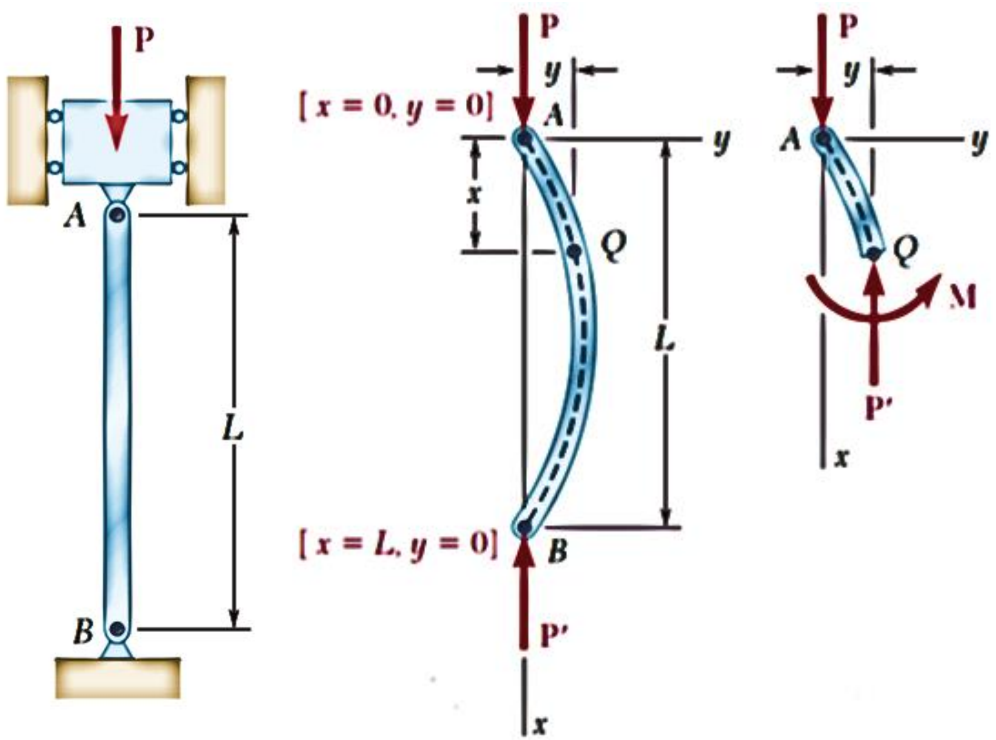




EULER FORMULA FOR PIN ENDED COLUMN

In 1759 a Swiss mathematician Leonhard Euler developed a theoretical analysis of premature failure due to buckling.

Let suppose a pin ended column **AB** of length ***L*** is subjected to a slight bending. Since column can be considered a beam placed in vertical direction and subjected to axial load, thus deformation at any point of column can be represented by equation of elastic curve.



$$EI\frac{d^2y}{dx^2} = M \qquad (1)$$

$$M = -P.y \qquad (2)$$

Here in figure, bending moment at point Q having co-ordinate (x , y) can be represent as given in Eqn. (2). The negative sign indicate the negative bending moment.

$$(1) \Rightarrow \quad EI\frac{d^2y}{dx^2} = -P.y$$

$$\frac{d^2y}{dx^2} + \frac{P.y}{EI} = 0 \qquad (3)$$

$$Let \qquad k^2 = \frac{P}{EI} \qquad (4)$$

$$(4) \Rightarrow \quad \frac{d^2y}{dx^2} + k^2y = 0 \qquad (5)$$

Eqn. (5) represent a second order *Homogeneous Differential Equation* for simple harmonic motion and general solution of the equation is given as Eqn. (6)

$$y = C\sin kx + D\cos kx \qquad (6)$$

Coefficient ***C*** & ***D*** can be determined by applying the boundary condition.

At End A: ***x = 0 & y = 0***

$$(6) \Rightarrow 0 = C\sin(k0) + D\cos(k0)$$

$$D = 0$$

At End B: ***x = L & y = 0***

$$(6) \Rightarrow 0 = C\sin kL + 0\cos kL$$

$$0 = C\sin kL \qquad (7)$$

In Eqn. (6) either ***A*** = 0 or sinkL=0. if ***A*** = 0 it will be zero everywhere along the column and we will have a trivial solution (member will be straight for any loading) the only

$$\sin kL = 0 \qquad (8)$$

To satisfy the Eqn. (8)

$$kL = n\pi \ (radian) \qquad (9)$$

n = 1, 2, 3, ….

$$(4) \Rightarrow \quad k = \sqrt{\frac{P}{EI}}$$

$$(9) \Rightarrow \quad \sqrt{\frac{P}{EI}}.L = n\pi$$

$$P = \frac{n^2\pi^2 EI}{L^2} \qquad (10)$$

n values of 1, 2, 3, represent the buckling shape (eigenvalue) corresponding to 1st, 2nd and 3rd buckling mode shape, respectively.

The smallest (critical) value load, P_{cr} occurs when n = 1, which corresponding to first (least) buckling mode.

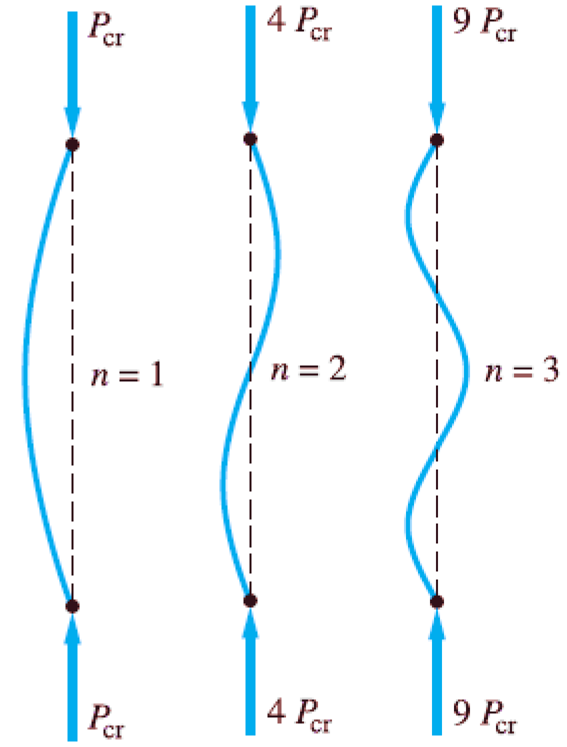


FIG. 10.4 First three buckling mode shapes of a simply supported column.

$$P_{cr} = \frac{\pi^2 EI}{L^2} \qquad (11)$$

The Eqn. (11) is called the Euler formula and deflection corresponding to this load is

$$(6) \Rightarrow \qquad y = C \sin kx = C \sin \sqrt{\frac{P_{cr}}{EI}}\,x \qquad (12)$$

Substituting the value of $\boldsymbol{P_{cr}}$ from Eqn. (11)

$$(12) \Rightarrow \quad y = A\sin\sqrt{\frac{\pi^2 EI}{EIL^2}}x = A\sin\frac{\pi.x}{L} \qquad (13)$$

Eqn. (13) represents the equation of elastic curve after the column has been buckled. From the equation (13) deflection will be maximum when

$$If \quad \sin\frac{\pi x}{L} = 1 \qquad (13) \Rightarrow \quad y_m = A$$

Above solution is indeterminate this is due to the fact that differential Eqn. (2) used is the linearized approximation of actual differential equation.

If $\boldsymbol{P} < \boldsymbol{P_{cr}}$ the condition $\sin(\pi x/L) = 0$ cannot be satisfied then we must have $C = 0$ as only in this case configuration of column will be straight, which is stable condition.

INFLUENCE OF END CONDITION

Effective Length (L_e)

It is the length of the column corresponding to the half sigh wave or length between the point of contra-flexure.

- The Euler critical load for fundamental buckling mode depends upon the effective length.

Effective Length Factor (K)

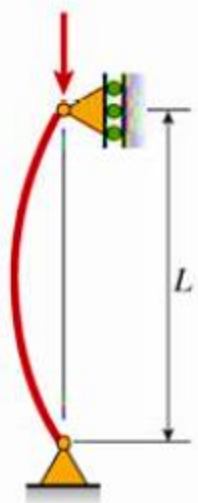
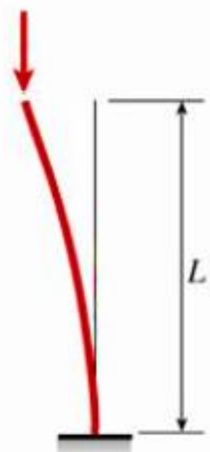
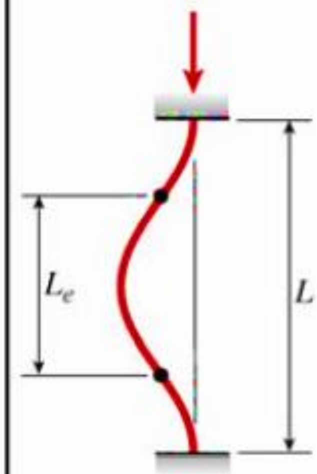
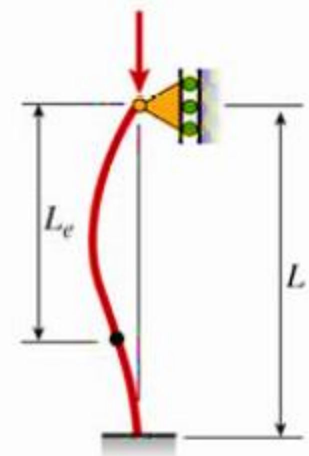
It is the ratio between the effective length and original length

$$K = \frac{L_e}{L}$$

$$\Rightarrow \quad L_e = KL$$

- The Factor K depends upon the end/boundary Condition of the column

Effect of K-factor on Critical Buckling Load

(a) Pinned-pinned column	(b) Fixed-free column	(c) Fixed-fixed column	(d) Fixed-pinned column
$P_{cr} = \dfrac{\pi^2 EI}{L^2}$	$P_{cr} = \dfrac{\pi^2 EI}{4L^2}$	$P_{cr} = \dfrac{4\pi^2 EI}{L^2}$	$P_{cr} = \dfrac{2.046\ \pi^2 EI}{L^2}$
			
$L_e = L$	$L_e = 2L$	$L_e = 0.5L$	$L_e = 0.699L$
$K = 1$	$K = 2$	$K = 0.5$	$K = 0.699$

Critical Stress (σ_{cr})

It is the stress corresponding to the *Euler Critical Load* and can be calculated as following.

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{L_e^{\,2} A} = \frac{\pi^2 E}{(L_e / r)^2} \qquad (14) \qquad \therefore I = Ar^2$$

Critical Stress (σ_{cr}) for Slender Column

The critical stress for slender columns may be fixed by dividing proportional or yield stress by factor of safety and corresponding limiting slenderness ratio can be determined by using the Eqn. (14).

$$Let \qquad \sigma_{cr} = \frac{\sigma_{pl}}{F.O.S.}$$

$$\sigma_{cr} = \frac{250}{1.25} = 200 \;\; MPa$$

$$(14) \Rightarrow \quad 200 = \frac{\pi^2\, 200 \times 10^3}{(L_e / r)^2}$$

$$\Rightarrow \quad L_e / r \approx 100$$

Alternatively

For slender columns, Length > 30(least X-sectional dimension)

Assuming a rectangular cross-section of ***bxh.***

$$r_{\min} = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{hb^3/12}{bh}} = \frac{b}{2\sqrt{3}}$$

$$\frac{L_e}{r_{\min}} = \frac{30b}{b/2\sqrt{3}} \approx 103$$

$$Let \qquad \frac{L_e}{r_{\min}} = 100$$

$$\sigma_{cr} = \frac{\pi^2 200\times10^3}{(100)^2} \approx 200\ \ MPa$$

Critical Stress (σ_{cr}) for Short Column

For Short columns critical stress is taken equal to the crushing or yield stress and slenderness ratio may be fixed by considering the, Length = 10 (least X-sectional dimension)

$$\frac{L_e}{r_{\min}} = \frac{10b}{b/2\sqrt{3}} \approx 34.6$$

$$Let \qquad \frac{L_e}{r_{\min}} = 30$$

EXAMPLE PROBLEM

A steel bar of rectangular cross section of **40x60** $\boldsymbol{mm^2}$ and pinned at the both ends is subjected to axial compression. If proportional limit of material is 230 MPa and E =200 GPa.

a) Determine the maximum length for which the Euler Equation may be used to calculate the buckling load.

b) For the same column determine the Euler Buckling load if length of the column is equal to 2m.

Data

Area = $\boldsymbol{bxh}$ = 40x60 mm2

σ_{Pl} = 230 MPa , E = 200 GPa , L = ?

b) P_{cr} = ? If L = 2 m

Example 10.01 *(Bear & Johnston 6th Ed.)*

A ***2.0 m*** long pin-ended column of square cross section is to be made of wood. Assuming $\boldsymbol{E}$ =13 GPa, $\boldsymbol{\sigma_{all}}$ = 12 MPa, and using a factor of safety of **2.5** in computing Euler's critical load for buckling, determine the size of the cross section if the column is to safely support.

a) A **100 *kN*** load

b) A **200 *kN*** load

Data

σ_{all} = 12 MPa , E = 13 GPa

F.O.S. = 2.5

L = 2.0 m

Size of square column, b = ?

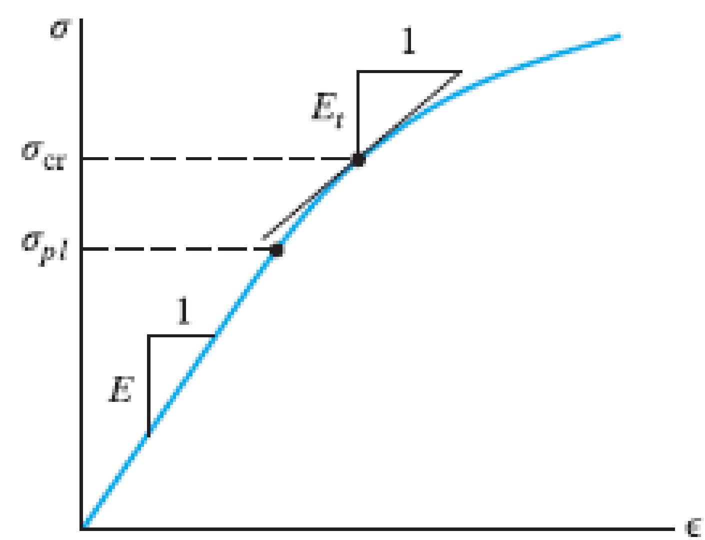
σ_{cr} FOR INTERMEDIATE COLUMNS

Tangent Modulus Theorem (Inelastic Buckling)

By this method a modified version of Euler equation is adopted to determine the stress-slenderness relationship in which the value of the modulus of elasticity at any given level.

Consider a column manufactured from the a material, whose stress-strain curve is shown in the figure below.

The slope of the tangent to the stress-strain curve at any stress value $\boldsymbol{\sigma}$ ($\boldsymbol{\sigma}$ is greater than $\boldsymbol{\sigma_{Pl}}$ and is within the inelastic range) is equal to the value of ***Tangent Modulus of Elasticity, E_t.***

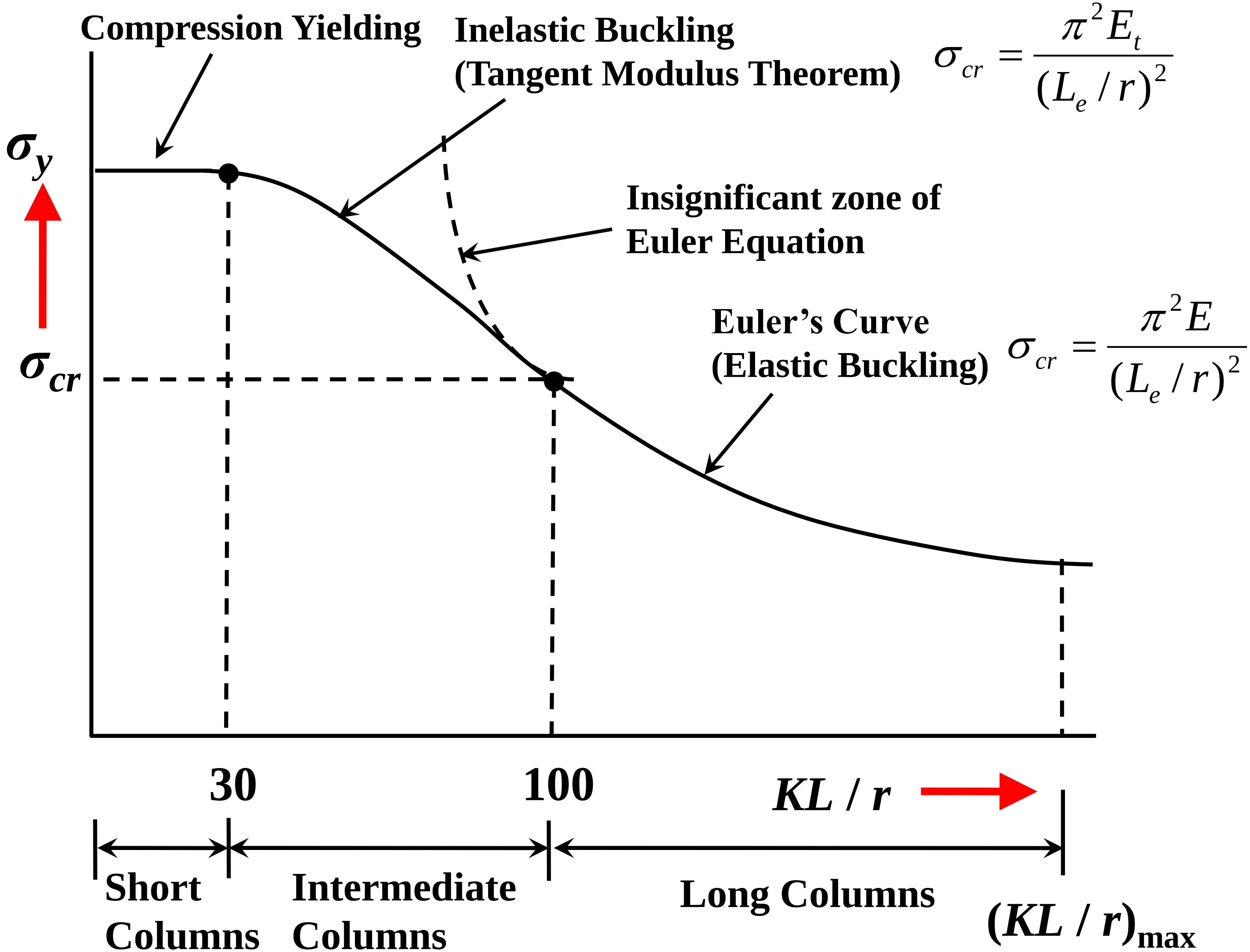


$\boldsymbol{E_t}$ is different from the $\boldsymbol{E}$ which is the value at Elastic limit.

- The value of $\boldsymbol{E_t}$ can be used is Euler equation to calculate the modified slenderness corresponding to any successive value of $\boldsymbol{\sigma}$.
- The curve for to intermediate column can be plotted by obtaining the slenderness value corresponding the any successive stress value ($\boldsymbol{\sigma = \sigma_{cr}}$) ranging between than $\boldsymbol{\sigma_{Pl}}$ and $\boldsymbol{\sigma_{ult}}$ or crushing value .

$$\sigma_{cr} = \frac{\pi^2 E_t}{(L_e / r)^2} \quad \Rightarrow \quad \frac{Le}{r} = \sqrt{\frac{\pi^2 E_t}{\sigma_{cr}}} \qquad (15)$$

- Although, the nonlinearity of the stress-strain diagram beyond the proportional limit is considered in Eqn. (15), its theoretical basis is somewhat weak. Therefore, this equation should be viewed as an empirical formula. However, the results obtained from Equation are in satisfactory agreement with experimental results.



Compression Yielding
Inelastic Buckling
(Tangent Modulus Theorem)
$\sigma_{cr} = \frac{\pi^2 E_t}{(L_e / r)^2}$
σ_y
Insignificant zone of
Euler Equation
Euler's Curve
(Elastic Buckling)
$\sigma_{cr} = \frac{\pi^2 E}{(L_e / r)^2}$
σ_{cr}
30
100
KL / r
Short
Columns
Intermediate
Columns
Long Columns
$(KL / r)_{max}$

Rankin-Gordon Formula

Euler formula is only suitable for the slender columns with small imperfections. In practice, most of the intermediate columns fail due to the combined effect of compression and flexure and experimentally obtained results are much less than the Euler prediction.

Gordon suggested an empirical formula based on the experimental results to predict the load of intermediate columns, which was further modified by Rankin.

According to Rankin intermediate columns/members fail due to buckling and compression to more or less degree and load carrying capacity of such member can be calculated as following.

$$\frac{1}{P_R} = \frac{1}{P_c} + \frac{1}{P_e} \Rightarrow P_R = \frac{P_c P_e}{P_c + P_e} = \frac{P_c}{1 + P_c / P_e} \qquad (16)$$

In Eqn. (16)

$\boldsymbol{P_R}$ = Rankin – Gordon buckling load

$\boldsymbol{P_e}$ = Euler buckling Load

$\boldsymbol{P_c}$ = Ultimate compressive load

For pin endded column

$$P_e = \frac{\pi^2 EI}{L_e^{\ 2}}$$

$$P_c = \sigma_c A \quad or \quad \sigma_y A$$

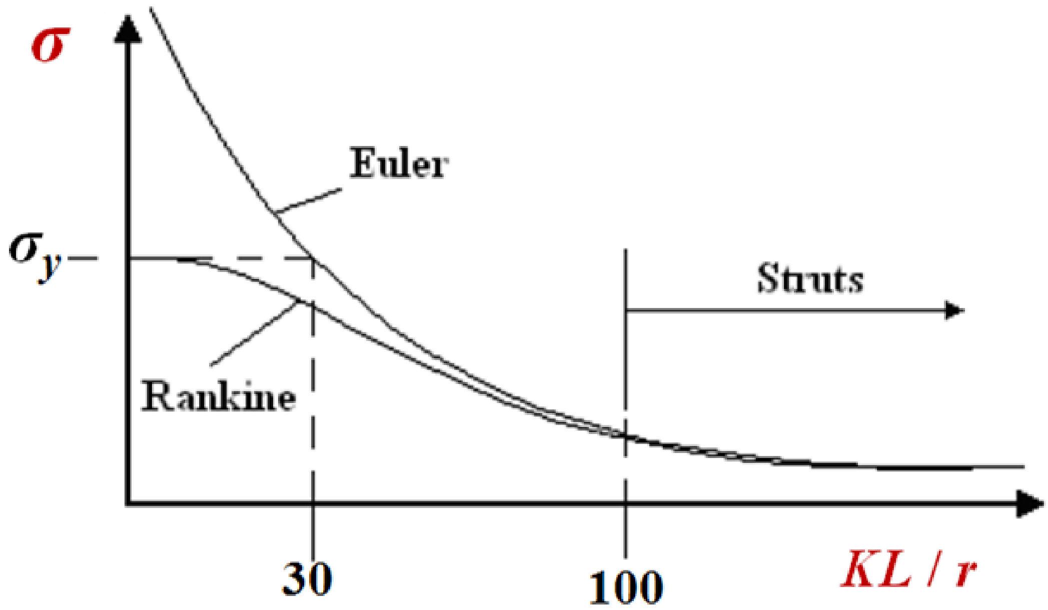
$$(16) \Rightarrow \qquad P_R = \frac{\sigma_y . A}{1 + \dfrac{\sigma_y . A}{\pi^2 EI / L_e^{\ 2}}} = \frac{\sigma_y . A}{1 + \dfrac{\sigma_y . A L_e^{\ 2}}{\pi^2 E (Ar^2)}} = \frac{\sigma_y . A}{1 + \dfrac{\sigma_y}{\pi^2 E} . \left(\dfrac{L_e}{r}\right)^2}$$

$$P_R = \frac{\sigma_y . A}{1 + a\left(\dfrac{L_e}{r}\right)^2} \qquad (17)$$

$$\therefore \qquad a = \frac{\sigma_y}{\pi^2 E}$$

a **=** Rankin constant, which depends upon the boundary condition and material properties

Graphical Presentation of Rankin Formula



Rankin constant for various Materials

Material	σ_y (MPa)	Rankin constant, a
Mild Steel	325	1 / 7500
Wrought Iron	250	1 / 9000
Cast Iron	560	1 / 1600
Timber	35	1 / 3000

Example Problem

A cast Iron column of 200 mm external diameter is 20 mm thick and 4.5 m long. Assuming the both end rigidly fixed, calculate the safe load using Rankin Formula if Rankin constant, $\boldsymbol{a}$ = 1 / 1600, $\boldsymbol{\sigma_y}$ = 550 *MPa* F.O.S. = 4.0.

Data

$\boldsymbol{\sigma_y}$ = 550 MPa ,

D_o = 200.0 mm

t = 20 mm

$\boldsymbol{P_{safe} = P_R / FOS}$

F.O.S. = 4.0

K = 0.5 (both Ends fixed)

a = 1 / 1600

$$P_R = \frac{\sigma_y . A}{1 + a\left(\frac{L_e}{r}\right)^2}$$

Practice Problem

For the given data determine the length of the Column for which Euler formula cease to apply.

Data

$\boldsymbol{\sigma_y}$ = 325 MPa ,

D_o = 38.0 mm

D_i = 33.0 mm

a = 1 / 7500

E = 200 GPa

L = 2.25 m

K = 1.0 (both Ends pinned)

AISC SPECIFICATIONS FOR STEEL COLUMNS

American Institute of Steel Construction (AISC) specifies two method for the computation of the compressive strength of the columns. Both design specification bound the maximum slenderness ratio equal to 200.

1. Allowable stress design (ASD)
2. Load and Resistance Factor Design (LRFD)

1.0 Allowable stress design (ASD)

It is the old method and according to this method columns made of structural steel can be designed on the basis of formulas proposed by the Structural Stability Research Council (SSRC). Factors of safety are applied to these formulas.

- It consider only intermediate (short) and long column and there is no straight portion between the stress~slenderness ratio curve. A specific slenderness ratio value $\boldsymbol{R_c}$ is used to differentiate between the slender and intermediate (or short) column.
- Experimental studies showed that compressive residual stresses can exist in rolled-formed steel sections their magnitude may be as much as one-half the yield stress. Consequently, if the stress in the Euler formula is greater $\boldsymbol{\sigma_y}$ **/2** then equation is not valid. Thus, *limiting slenderness ratio* $\boldsymbol{R_c}$ for the long columns can be determined by putting the $\boldsymbol{\sigma_{cr} = \sigma_y /2}$ in Euler Equation.

$$\frac{\sigma_y}{2} = \frac{\pi^2 E}{(L_e / r)^2} \quad \Rightarrow \quad R_c = \frac{L_e}{r} = \sqrt{\frac{\pi^2 E}{\sigma_y}} \qquad (18)$$

Slender Column

$$If \quad 200 \geq \frac{L_e}{r} \geq R_c \quad Its\ \ long\ column$$

In long column allowable stress can be calculated through the Euler equation divided by the Factor of safety.

$$\sigma_{all} = \frac{\sigma_{cr}}{FOS} = \left(\frac{\pi^2 E}{(L_e / r)^2} \right) \frac{1}{FOS} \quad \Rightarrow \quad \frac{12}{23} . \frac{\pi^2 E}{(L_e / r)^2} \qquad (19)$$

Short Column

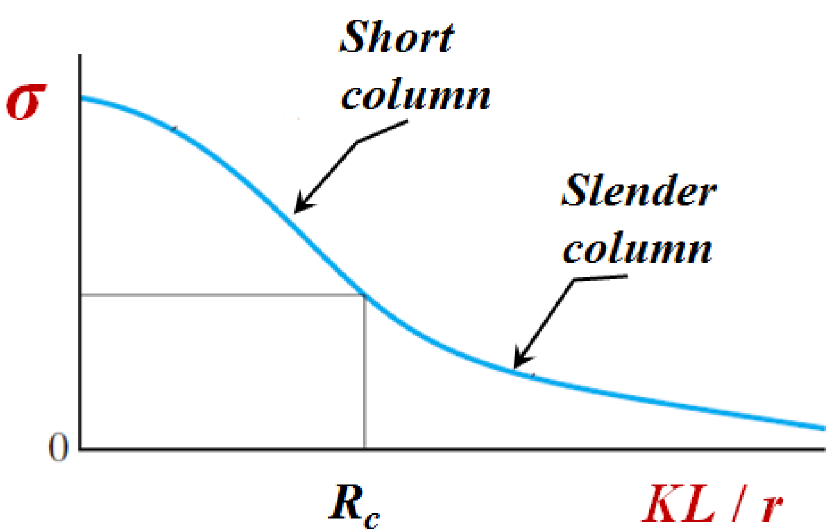
$$If \quad \frac{L_e}{r} < R_c \quad Its\ \ short / Intermedioate\ column$$

The short column are designed on the base of an empirical formula which is parabolic in form and maximum stress by this formula is given as following.

$$\sigma_{\max} = \left(1 - \frac{(L_e / r)^2}{2R_c^{\ 2}}\right)\sigma_y \qquad (20)$$

$$FOS = \frac{5}{3} + \frac{3}{8}\left(\frac{L_e / r}{R_c}\right) - \frac{1}{8}\frac{(L_e / r)^3}{R_c^{\ 3}} \qquad (21)$$

$$\sigma_{all} = \frac{\sigma_{\max}}{FOS} \qquad (22)$$



FOS becomes 5/3 or 1.67 when $L_e/r = 0$ and increases to 1.92 or 23/12 at slenderness value equal to $\boldsymbol{R_c}$.

All the above equation may be used both in SI and FPS System.

Example Problem

Determine the allowable axial load (P_{allow}) for a W310x129 wide-flange steel column with both end pinned, for the following lengths.

(a) L = 6 m (b) L = 9 m

Assume E = 200 GPa and σ_y = 340 MPa

Data

For W310x129 Section

A= 16,500 mm^2

σ_y = 340 MPa

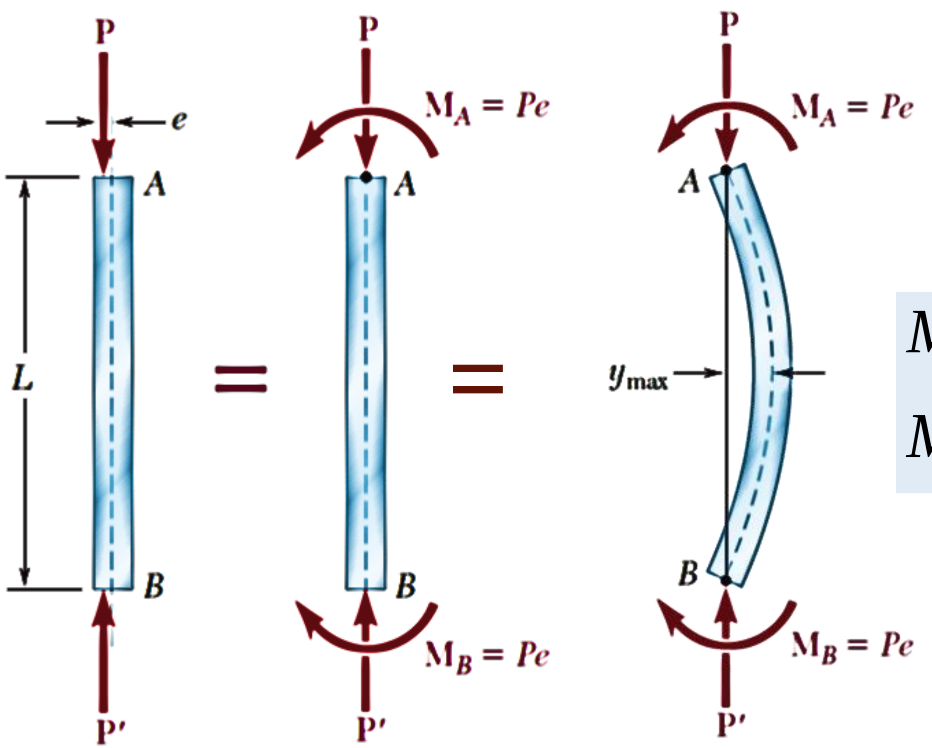
$r_z = r_y$ = 78.0 mm

E = 200 GPa

K = 1.0 (both Ends pinned)

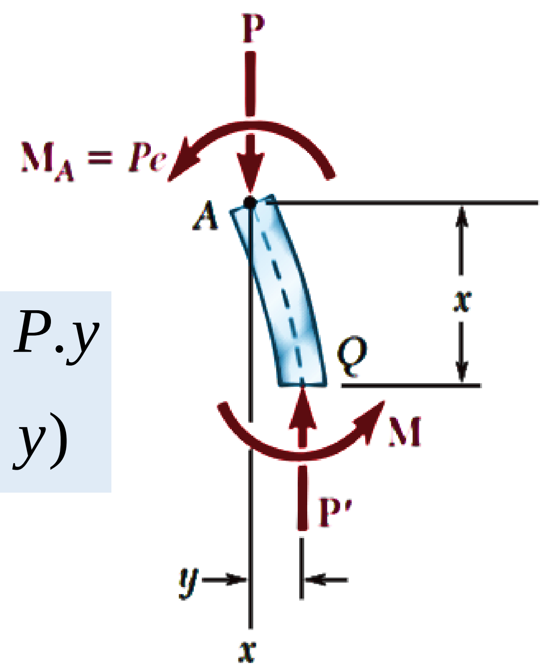
ECCENTRICALLY LOADED COLUMN (SECANT FORMULA)

In practice it is difficult to apply the end thrust (axial load) along the longitudinal centroidal axes of columns. In such case we have to consider the effect of eccentrically applied load "***P***" on a prismatic column of *flexural stiffness* ***EI***.



$$M_Q = M_A + P.y$$

$$M_Q = P(e + y)$$



$$EI\frac{d^2y}{dx^2} = M \qquad (1)$$

$$M_Q = -P(e+y) \qquad (23)$$

Suppose axial load is acting at an eccentricity "e" from the weaker axes (y-axis) the equation of elastic curve and moment at any arbitrary point Q can be given in Eqn. (23).

$$(1) \Rightarrow \quad EI\frac{d^2y}{dx^2} = -P(e+y)$$

$$\frac{d^2y}{dx^2} + \frac{P.y}{EI} = -\frac{P.e}{EI} \qquad (24)$$

$$Let \qquad k^2 = \frac{P}{EI} \qquad (4)$$

$$(24) \Rightarrow \quad \frac{d^2y}{dx^2} + k^2y = -k^2e \qquad (25)$$

The complete solution of Eqn. (25) is given as following

$$y = \underbrace{C\sin kx + D\cos kx}_{\textbf{General solution}}\ \underbrace{-\,e}_{\textbf{Particular solution}} \qquad (26)$$

Coefficient ***C*** & ***D*** can be determined by applying the boundary condition.

At End A: ***x = 0 & y = 0***

$$(26) \Rightarrow 0 = C\sin(k0) + D\cos(k0) - e \quad \Rightarrow \quad D = e$$

At End B: ***x = L & y = 0***

$$(26) \Rightarrow 0 = C\sin kL + e\cos kL - e$$

$$C\sin kL = e(1-\cos kL)$$

$$C\left(2\sin\frac{kL}{2}\cos\frac{kL}{2}\right) = e\left(2\sin^2\frac{kL}{2}\right)$$

$$\therefore \sin kL = 2\sin\frac{kL}{2}\cos\frac{kL}{2}$$

$$\therefore (1-\cos kL) = 2\sin^2\frac{kL}{2}$$

$$C = e\tan\frac{kL}{2}$$

Substituting the value ***C*** & ***D*** in Eqn. (26)

$$y = e\left[\tan\frac{kL}{2}\sin kx + \cos kx - 1\right] \qquad (27)$$

The Eqn. (27) represents the equation of deflection (***y***) at any point (***x***) along the columns. The value of maximum deflection (y_{max}) can be calculated by setting $x = L/2$.

$$(27) \Rightarrow \qquad y_{\max} = e\left[\tan\frac{kL}{2}\sin\frac{kL}{2} + \cos\frac{kL}{2} - 1\right]$$

$$y_{\max} = e\left[\frac{\sin^2 kL/2 + \cos^2 kL/2}{\cos kL/2} - 1\right]$$

$$y_{\max} = e\left[\sec\frac{kL}{2} - 1\right] \qquad (28)$$

$$(4) \Rightarrow \qquad \therefore k = \sqrt{\frac{P}{EI}}$$

$$y_{\max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\,\frac{L}{2}\right) - 1\right] \qquad (29)$$

The Eqn. (29) shows that (y_{max}) becomes infinite when $P = P_{cr}$.

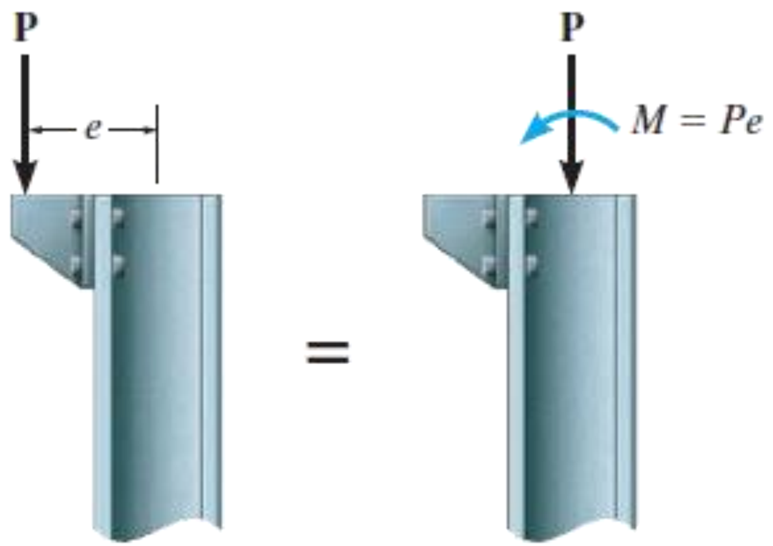
$$y_{\max} = e\left[\sec\left(\sqrt{\frac{\pi^2 EI}{EIL^2}}\,\frac{L}{2}\right) - 1\right]$$

$$y_{\max} = e\left[\sec\frac{\pi}{2} - 1\right]$$

$$\therefore \qquad \sec\frac{\pi}{2} = \infty$$

In actual cases deflection does not become infinite even the load exceed the elastic limits also ***P*** should not be reached to the P_{cr} (Euler critical load)

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad \Rightarrow \quad EI = \frac{P_{cr} L^2}{\pi^2}$$



Source of eccentricity in column

Replacing the value of ***EI*** in Eqn. (29)

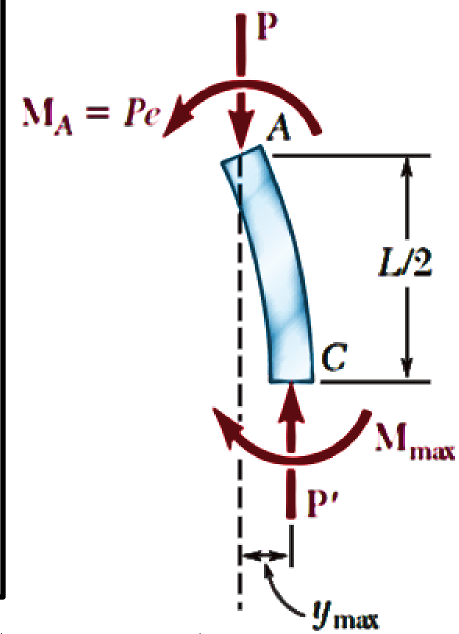
$$y_{max} = e\left[\sec\left(\sqrt{\frac{P\pi^2}{P_{cr} L^2}}\frac{L}{2}\right) - 1\right]$$

$$y_{max} = e\left[\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right] \qquad (30)$$

Note: In above equation secant angle is in radians

MAXIMUM STRESS IN ECCENTRIC COLUMN

The maximum stress σ_{max} occurs in the section of the column where the bending moment is maximum, ***i.e.***, in the transverse section through the midpoint ***C***, and can be obtained by adding the normal stresses due to the axial force and the bending couple exerted on that section



$$\sigma_{max} = \frac{P}{A} + \frac{M_{max} \times c}{I} \qquad (31)$$

$$\sigma_{max} = \frac{P}{A} + \frac{P(y_{max} + e) \times c}{Ar^2}$$

$$\therefore M_C = P(y_{max} + e)$$

$$\therefore y_{max} = e\left[\sec\frac{kL}{2} - 1\right]$$

$$OR \qquad y_{max} = e\left[\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right]$$

$$\sigma_{\max} = \frac{P}{A} + \frac{P(y_{\max} + e)\times c}{Ar^2}$$

$$\sigma_{\max} = \frac{P}{A}\left[1 + \left\{e\left(\sec\frac{kL}{2} - 1\right) + e\right\}\frac{c}{r^2}\right]$$

$$\sigma_{\max} = \frac{P}{A}\left[1 + \left\{\sec\frac{kL}{2} - 1 + 1\right\}\frac{ec}{r^2}\right]$$

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\frac{kL}{2}\right] \qquad (32)$$

e = eccentricity of loading
c = distance from the N.A. to extreme fibres
r = radius of gyration
A = cross-sectional area of column

Replacing the value of ***kL /2*** as following

$$\frac{kL}{2} = \sqrt{\frac{P}{EI}}\frac{L}{2} = \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}$$

$$(32) \Rightarrow \qquad \sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right] \qquad (33)$$

- The Eqn. (33) can be used for any end condition as long as the appropriate (***K***) value is used to calculate P_{cr}.
- Since $\sigma_{\max}$ does not vary linearly with load ***P***, the principal of superposition is not applicable to determine the stress due to the simultaneously application of applied loads.
- For the same reason any factor of safety should be used with load not the stress.

$$(32) \Rightarrow \qquad \sigma_{\max} = \frac{P}{A}\left[1+\frac{ec}{r^2}\sec\sqrt{\frac{P}{EI}}\frac{L}{2}\right]$$

$$\therefore \qquad I = Ar^2$$

$$\sigma_{\max} = \frac{P}{A}\left[1+\frac{ec}{r^2}\sec\sqrt{\frac{P}{EAr^2}}\frac{L_e}{2}\right]$$

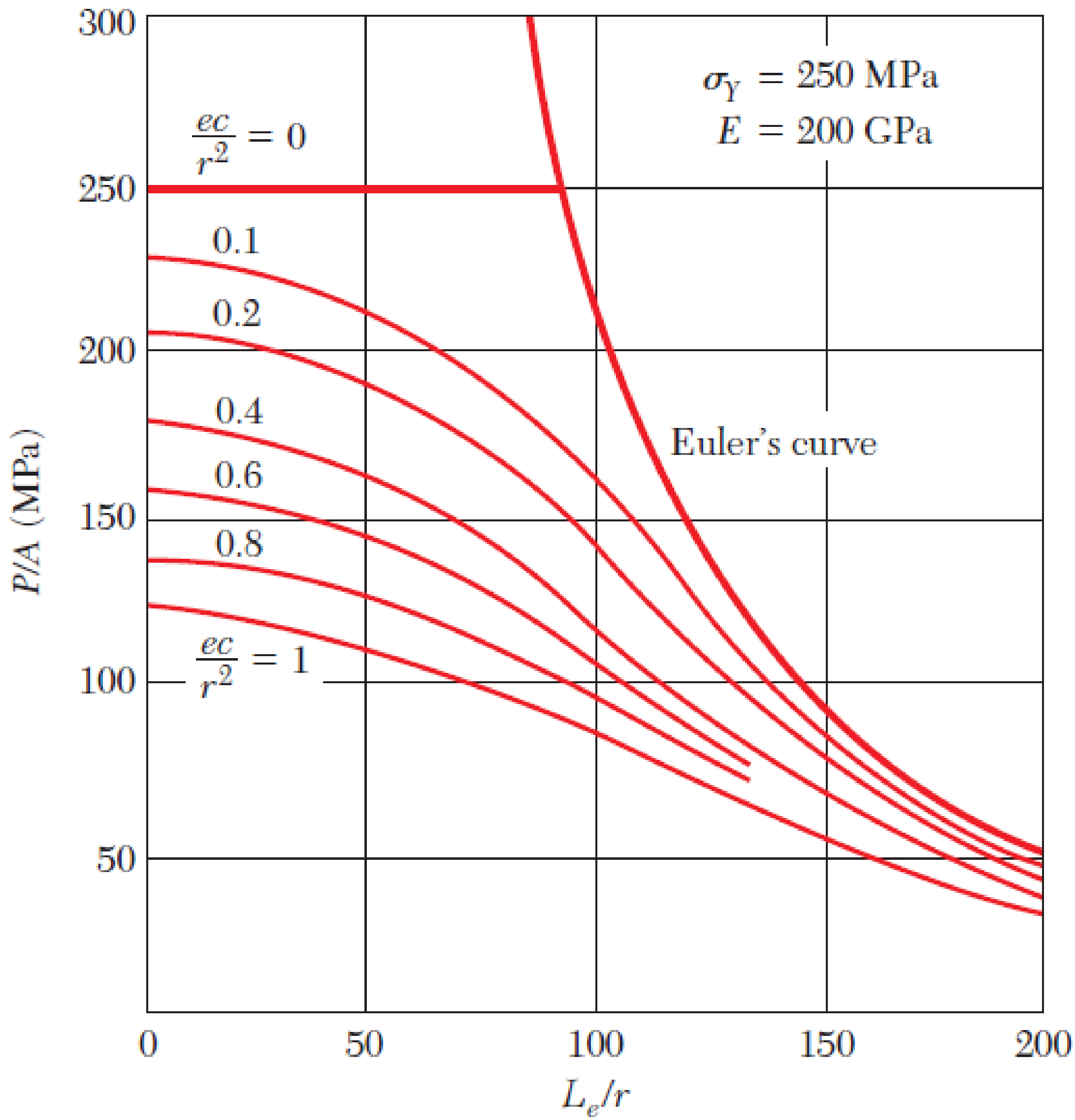
$$\frac{ec}{r^2} = Eccentricity\ ratio$$

$$\frac{P}{A} = \frac{\sigma_{\max}}{\left[1+\frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{L_e}{r}\right)\right]} \qquad (34)$$

$$\sqrt{\frac{P}{EAr^2}}\frac{L_e}{2} = Euler\ angle$$

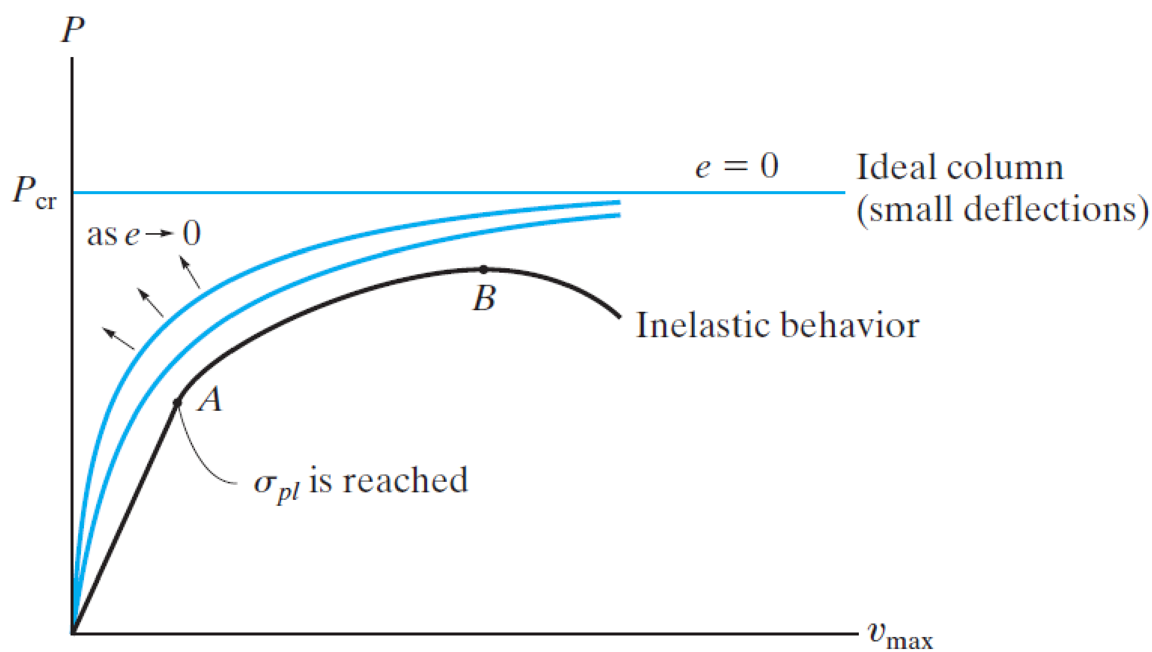
The formula given in Eqn. (33) is referred to as ***the secant formula***; it defines the force per unit area (P/A), that causes a specified maximum stress (σ_{max}) in a column of given effective slenderness ratio (L_e/r), for a given value of the eccentricity ratio (ec/r^2).

- If the material properties, the dimensions of the column, and the eccentricity e are known then we have two variables in the secant formula: P and σ_{max}. If P is also given, σ_{max} can be computed from the formula without difficulty.
- On the other hand, if σ_{max} is specified, the determination of P is considerably more complicated because Eqn. (33), being nonlinear in P, must be solved by trial-and-error.
- The secant formula is chiefly useful for intermediate values of L_e/r. However, to use it effectively, we should know the value of the eccentricity e of the loading



300
250
200
150
100
50
$\frac{ec}{r^2} = 0$
0.1
0.2
0.4
0.6
0.8
$\frac{ec}{r^2} = 1$
$\sigma_Y = 250$ MPa
$E = 200$ GPa
Euler's curve
P/A (MPa)
0
50
100
150
200
L_e/r

- Due to imperfections in manufacturing or specific application of the load, a column will never suddenly buckle; instead, it begins to bend.
- The load applied to a column is related to its deflection in a nonlinear manner, and so the principle of superposition does not apply.
- As the slenderness ratio increases, eccentrically loaded columns tend to fail at or near the Euler buckling load.



Exercise: Plot the load-displacement curves of a rectangular column for the given data with eccentricity ranging from 5-25 mm.

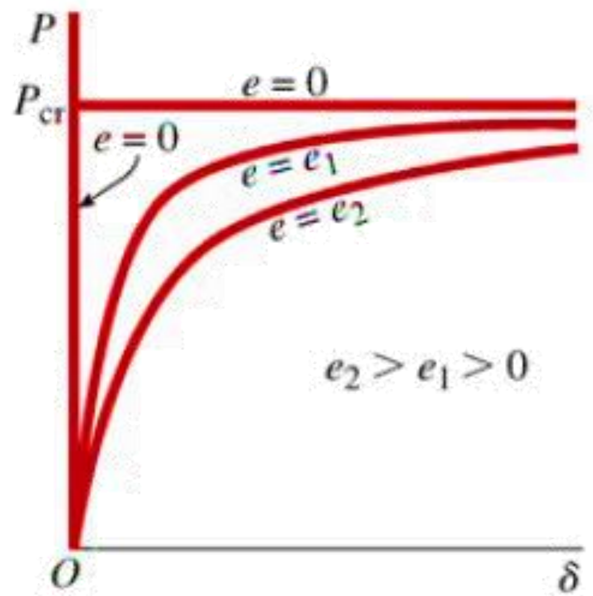
Data

L= 2.5 m $\quad$ K = 1.0, $\quad$ A = 30x60 mm^2, $\quad$ e = 5 – 25 mm

Solution $\quad$ $I_{min} = 60x30^3/12 = 135{,}000\ mm^4$, $r_{min} = 8.66\ mm$

$$P_{cr} = \frac{\pi^2 \times 200 \times 10^3 \times 135 \times 10^3}{(1 \times 2500)^2} 42.64kN \qquad y_{\max} = e\left[\sec\frac{kL}{2} - 1\right] = e\left[\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right]$$

P (*kN*)	**Sec(*kL*/2)**	***y* (mm)**				
0		***e*= 5**	***e*= 10**	***e*= 15**	***e*= 20**	***e*= 25**
10						
20						
30						
40						
42.64						



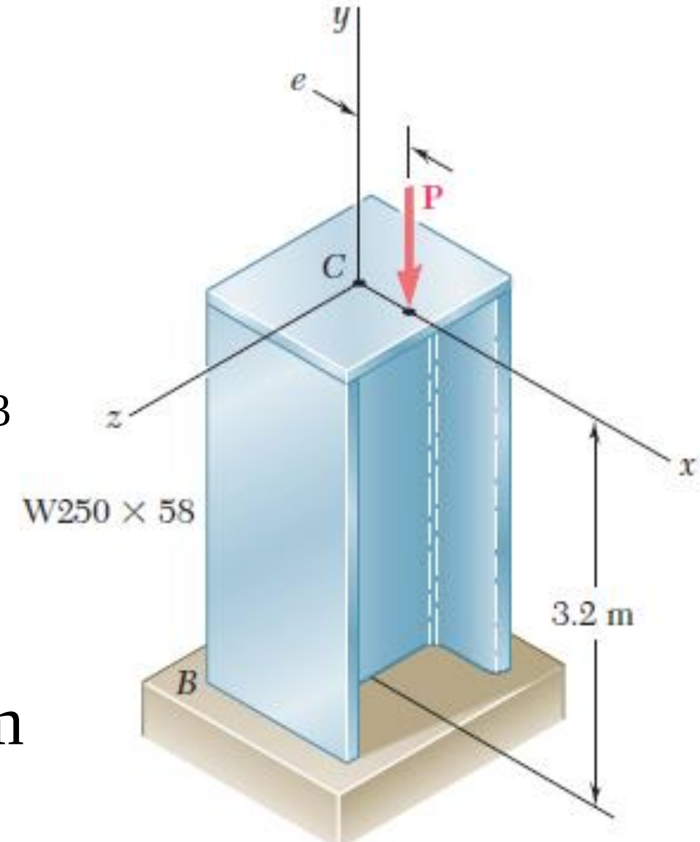
<u>Problem</u> 10.34 (Mech. of Materials, Bear & Johnston, 6th Ed)

The axial load ***P*** is applied at a point located on the *x*-axis at a distance ***e*** from the geometric axis of the rolled-steel column *BC*.

When P = 350 kN, the horizontal deflection of the top of the column is 5 mm. Using E =200 GPa. determine

(a) the eccentricity ***e*** of the load,

(b) the maximum stress in the column.



Data

<u>For W250x58 Section</u>

$A = 7420\ mm^2$ $\qquad S_y = 185\text{x}10^3\ mm^3$

$I_x = 8700\text{x}10^4\ mm^4$, $\ r_x = 108\ mm$

$I_y = 1870\text{x}10^4\ mm^4$, $\ r_y = 50.3\ mm$

E = 200 GPa, d = 250 mm, b_f =203 mm

K = 2.0 (cantilever case)

Sample Problem 10.3

(Mech. of Materials, A Pytel, 2nd Ed)

A W14x61 section is used as a simply supported column of 25 ft long. When the 150-kip load is applied with the 4-in. eccentricity shown, Determine

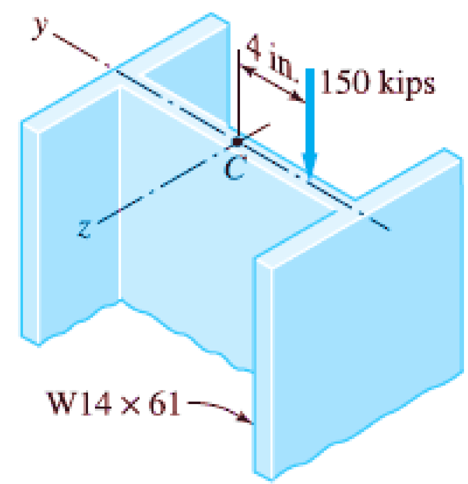
(1) the maximum compressive stress in the column;

(2) the factor of safety against yielding; and

(3) The maximum lateral deflection.

Assume that the column does not buckle about the y-axis.

Use E = 29x10^6 psi and $\boldsymbol{\sigma_y}$ = 36x10^3 psi..



For W250x58 Section

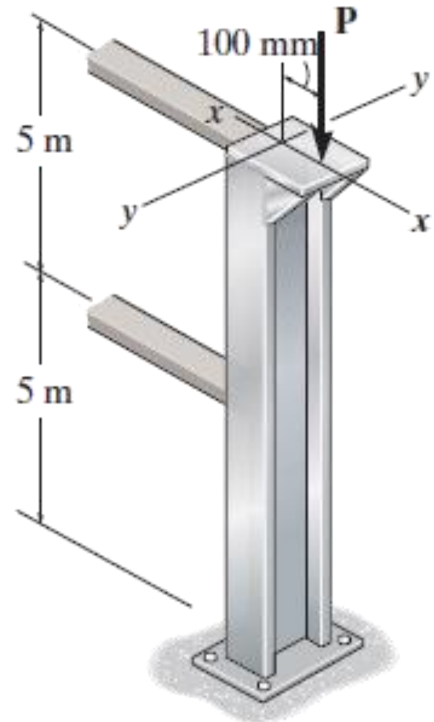
A= 17.9 in^2 $\quad$ I_z = 640 in^4 ,

S_z = 92.1 mm $\quad$ e = 4 in ,

Problem 13.53

(Mech. of Materials by RC Hibbler, 8^{th} Ed)

The W200x22, A-36-steel column is fixed at its base. Its top is constrained to rotate about the *y*–*y* axis and free to move along the *y*–*y* axis. Also, the column is braced along the *x*–*x* axis at its mid-height. Determine the allowable eccentric force *P* that can be applied without causing the column either to buckle or yield. Use against buckling F.O.S. = 2.0 and F.O.S. = 1.5 against yielding.



For W250x58 Section

$A = 28600\ mm^2$	$I_x = 20 \times 10^6\ mm^4$,	$r_x = 83.6\ mm$
$e = 100\ mm$,	$I_y = 1.42 \times 10^6\ mm^4$,	$r_y = 22.3\ mm$
$d = 200\ mm$,	$b_f = 102\ mm$,	$E = 200\ GPa$

INITIALLY CURVED COLUMN (PERRY - ROBERTSON FORMULA)

- In practice a column cannot be made perfectly straight and P_{cr} is never reached. Consideration of small deviation from the straight configuration makes the analysis more realistic.
- According to Perry-Robertson Formula, all practical imperfections (e.g. properties of the real columns) could be represented by a hypothetical initial curvature (a_0) of column.

Let consider a columns **AB** of length L has an initial imperfection y_0 prior to the application of the and y is the additional deformation due to the applied load P. the equation of the elastic curve for any arbitrary point Q can be represented as following.

$$EI\frac{d^2y}{dx^2} = M \qquad (1)$$

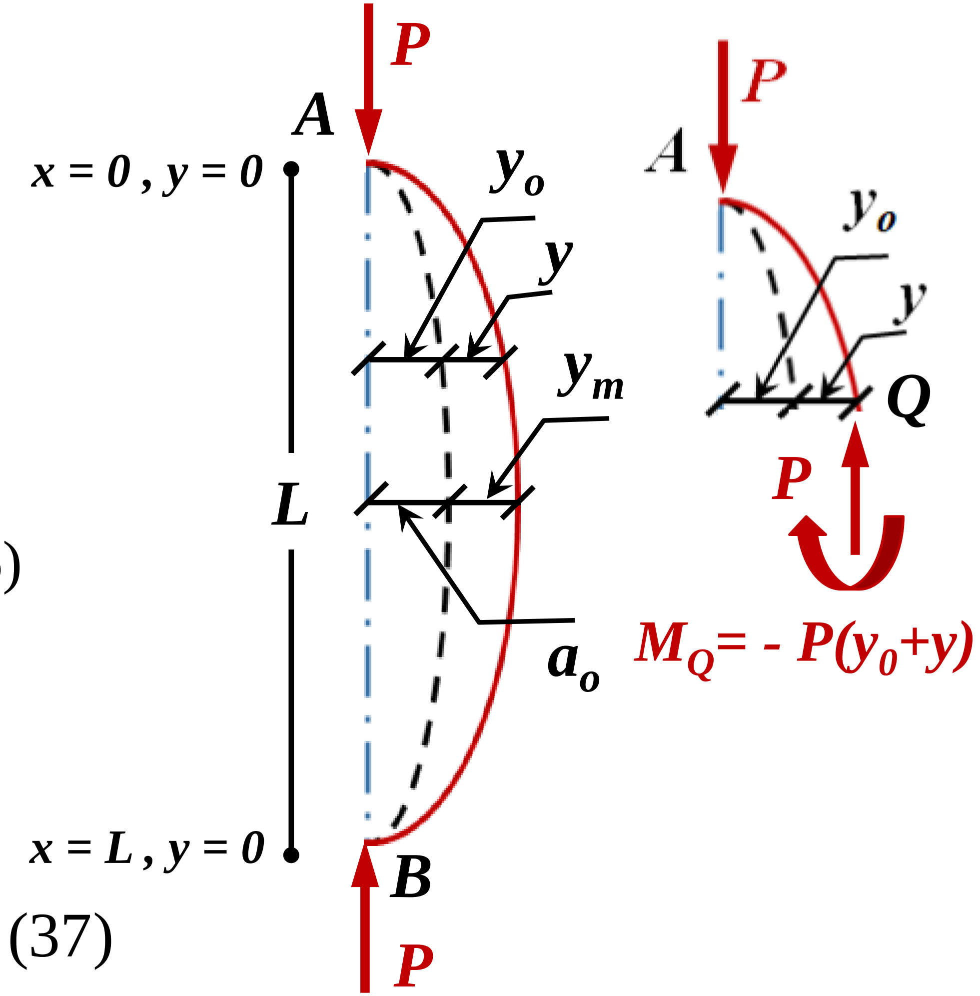
$$M_Q = -P(y_0 + y) \qquad (35)$$

$$(1) \Rightarrow \quad EI\frac{d^2y}{dx^2} = -P(y_0 + y)$$

$$\frac{d^2y}{dx^2} + \frac{P.y}{EI} = -\frac{P.y_0}{EI} \qquad (36)$$

$$Let \qquad k^2 = \frac{P}{EI} \qquad (4)$$

$$(36) \Rightarrow \quad \frac{d^2y}{dx^2} + k^2y = -k^2y_0 \qquad (37)$$



y_0 = initial deviation of the column and is represented by the sinusoidal curve

$$y_0 = a_0 \sin\frac{\pi x}{L} \qquad (38)$$

$$(37) \Rightarrow \quad \frac{d^2y}{dx^2} + k^2 y = -k^2 a_0 \sin\frac{\pi x}{L} \qquad (38)$$

The complete solution of Eqn. (38) is given as following

$$y = C\sin kx + D\cos kx - \frac{k^2 a_0}{\left(\frac{\pi^2}{L^2} - k^2\right)}\sin\frac{\pi x}{L} \qquad (39)$$

Applying the boundary condition

At End A: ***x = 0 & y = 0***

$$(39) \Rightarrow 0 = C\sin(k0) + D\cos(k0) - 0$$

$$\Rightarrow \quad D = 0$$

At End B: ***x = L & y = 0***

$$0 = C\sin kL + 0\cos kL - \frac{k^2 a_0}{\left(\frac{\pi^2}{L^2} - k^2\right)}\sin\frac{\pi L}{L}$$

In Eqn. (40) either C or sinkL is zero

$$0 = C\sin kL \qquad (40)$$

Assuming ***k*** any non-zero value (as deflection will always be due to some applied load ***P***) we must have **C = 0**

Substituting the values of C and D in Eqn. (39)

$$(39) \Rightarrow \quad y = \frac{k^2 a_0}{\left(\frac{\pi^2}{L^2} - k^2\right)} \sin\frac{\pi x}{L} = \frac{a_0}{\left(\frac{\pi^2}{k^2 L^2} - 1\right)} \sin\frac{\pi x}{L}$$

$$y = \frac{a_0}{\left(\frac{\pi^2 EI}{L^2 P} - 1\right)} \sin\frac{\pi x}{L} = \frac{a_0}{\left(\frac{P_{cr}}{P} - 1\right)} \sin\frac{\pi x}{L} \qquad (41)$$

$$\therefore \quad k^2 = \frac{P}{EI}$$

$$\therefore \quad P_{cr} = \frac{\pi^2 EI}{L^2}$$

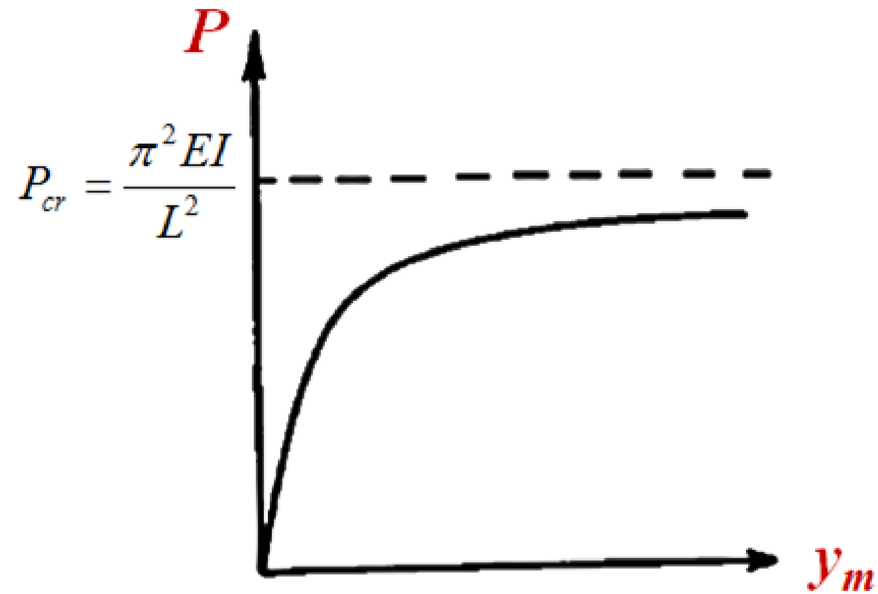
For pin ended column the deflection is maximum (***y_m***) at center when ***x = L/2***

$$(41) \Rightarrow \quad y_m = \frac{a_0}{\left(\frac{P_{cr}}{P} - 1\right)} \sin\frac{\pi (L/2)}{L} = \frac{a_0}{\left(\frac{P_{cr}}{P} - 1\right)} \qquad (42)$$

In Eqn. (41) & (42) ***y*** and ***y_m*** are the additional deflection due to the applied ***P*** as compared to the initial deflection ***a_0***.

Using Eqn. (42)

$$\begin{aligned}
&if\ P = 0.2P_{cr} &&\Rightarrow\ y_m = 0.25a_0\\
&if\ P = 0.5P_{cr} &&\Rightarrow\ y_m = a_0\\
&if\ P = 0.75P_{cr} &&\Rightarrow\ y_m = 3a_0\\
&if\ P = 0.9P_{cr} &&\Rightarrow\ y_m = 9a_0\\
&if\ P = P_{cr} &&\Rightarrow\ y_m = \infty
\end{aligned}$$



Load-deflection Curve of initially curved column

- The relationship of ***P*** and y_m as shown in the figure depicts that the initially deformed columns fails before reaching the P_{cr} (Euler critical load) and y_m increases rapidly with the increase of load ***P***.
- At any definite displacement before the failure the Eqn. (42) be written as following.

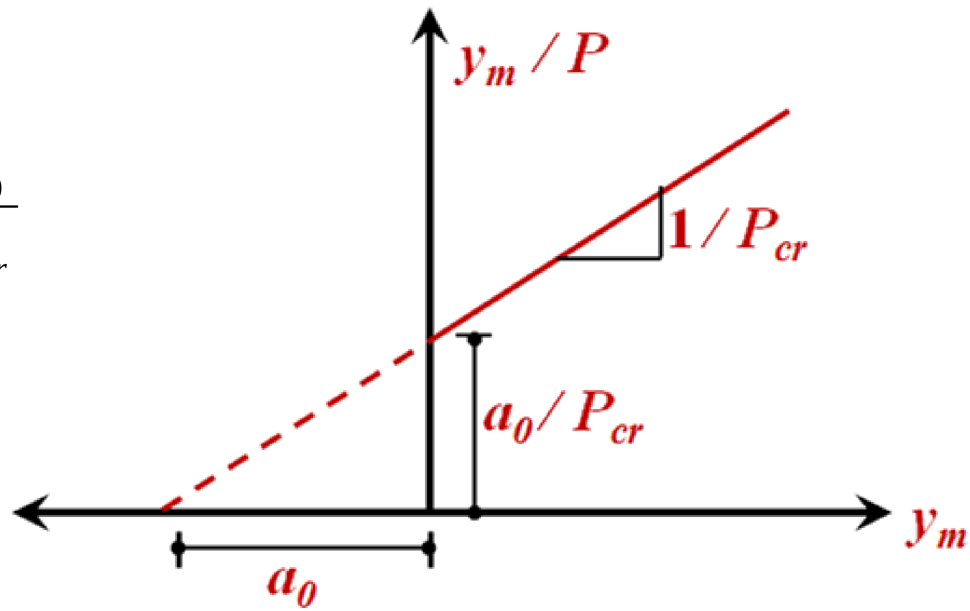
$$(42) \Rightarrow y_m\left(\frac{P_{cr}}{P}-1\right)=a_0$$

$$y_m\frac{P_{cr}}{P}-y_m=a_0 \quad \Rightarrow \quad \frac{P_{cr}}{P}=\frac{y_m}{P_{cr}}+\frac{a_0}{P_{cr}}$$

Relating with the equation of straight Line

$$y \quad = \quad mx \quad + \quad c$$

$$\left(\frac{P_{cr}}{P}\right)=\left(\frac{1}{P_{cr}}\right)y_m+\left(\frac{a_0}{P_{cr}}\right) \qquad (43)$$



South-Well Plot

- The values of y_m /P and y_m are plotted from a column test then these variables can be related by a straight line.
- While plotting initial values may be discarded (40% to 80% data may be plotted).
- This plot is called the South-well plot and it is used to determine the initial deflection of a column, experimentally.

Total deflection at any distance ***x*** is given as

$$y_t = y + y_0 \quad = \quad a_0 \sin\frac{\pi x}{L} + \frac{a_0}{\left(\frac{P_{cr}}{P} - 1\right)} \sin\frac{\pi x}{L} \quad = \quad a_0 \sin\frac{\pi x}{L}\left(\frac{1}{\frac{P_{cr}}{P} - 1} + 1\right)$$

$$y_t = a_0 \sin\frac{\pi x}{L}\left(\frac{P}{P_{cr} - P} + 1\right) \quad = \quad a_0 \sin\frac{\pi x}{L}\left(\frac{P_{cr}}{P_{cr} - P}\right)$$

$$y_t = a_0 \sin\frac{\pi x}{L}\left(\frac{P_{cr}}{P_{cr} - P}\right)\frac{A}{A} \quad = \quad a_0 \sin\frac{\pi x}{L}\left(\frac{\sigma_{cr}}{\sigma_{cr} - \sigma}\right) \qquad (44)$$

Displacement will be maximum at ***x = L /2***

$$\left(y_t\right)_{\max} \quad = \quad a_0 \sin\frac{\pi (L/2)}{L}\left(\frac{\sigma_{cr}}{\sigma_{cr} - \sigma}\right) \quad = \quad a_0\left(\frac{\sigma_{cr}}{\sigma_{cr} - \sigma}\right) \qquad (45)$$

MAXIMUM STRESS IN DEFLECTED COLUMN

The maximum stress σ_{max} occurs in the section of the column where the bending moment or displacement is maximum.

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max} \times c}{I} \qquad (46)$$

$$\sigma_{max} = \frac{P}{A} + \frac{P(y_t)_{max} \times c}{Ar^2}$$

$$\sigma_{max} = \frac{P}{A} + \frac{Pa_0\left(\frac{\sigma_{cr}}{\sigma_{cr} - \sigma}\right) \times c}{Ar^2}$$

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{a_0 c}{r^2}\left(\frac{\sigma_{cr}}{\sigma_{cr} - \sigma}\right)\right]$$

$$\sigma_{max} = \sigma\left[1 + \eta\left(\frac{\sigma_{cr}}{\sigma_{cr} - \sigma}\right)\right] \qquad (47)$$

$$\therefore \quad \eta = \frac{a_0 c}{r^2} = Initial\ deflection\ ratio$$

$$\therefore \quad \sigma = averge\ applied\ stress$$

$$\therefore \quad \sigma_{cr} = Euler\ critical\ stress$$

If applied load P is given the maximum stress can be determined by using the Eqn. (47)

If σ_{max} are specified then to determine the safe applied load the Eqn. (47) is to transformed in term of applied stress σ.

$$(47) \Rightarrow \sigma_{\max} = \sigma\left[1+\eta\left(\frac{\sigma_{cr}}{\sigma_{cr}-\sigma}\right)\right] = \sigma\left(\frac{\sigma_{cr}-\sigma+\sigma_{cr}\eta}{\sigma_{cr}-\sigma}\right)$$

$$\sigma_{\max}.(\sigma_{cr}-\sigma) = \sigma\sigma_{cr} - \sigma^2 + \sigma\sigma_{cr}\eta$$

$$\sigma^2 - \sigma(\sigma_{\max}+\sigma_{cr}+\sigma_{cr}\eta) + \sigma_{\max}\sigma_{cr} = \sigma^2 - \sigma[\sigma_{\max}+(1+\sigma_{cr})\eta] + \sigma_{\max}\sigma_{cr} = 0$$

$$\sigma = \frac{1}{2}[\sigma_{\max}+(1+\eta)\sigma_{cr}] - \sqrt{\frac{1}{4}[\sigma_{\max}+(1+\eta)\sigma_{cr}]^2 - \sigma_{\max}\sigma_{cr}} \qquad (48)$$

- We need not to consider positive square root since we are only interested in smaller values of square roots in the Eqn. (48).
- This equation represents the average value of stress in the cross-section at which the maximum stress would be attained at mid-height of the column for any given value of η.

To determine the average applied stress (**σ**) at which yield occurs then **σ_{max}** is replaces by the **σ_y**.

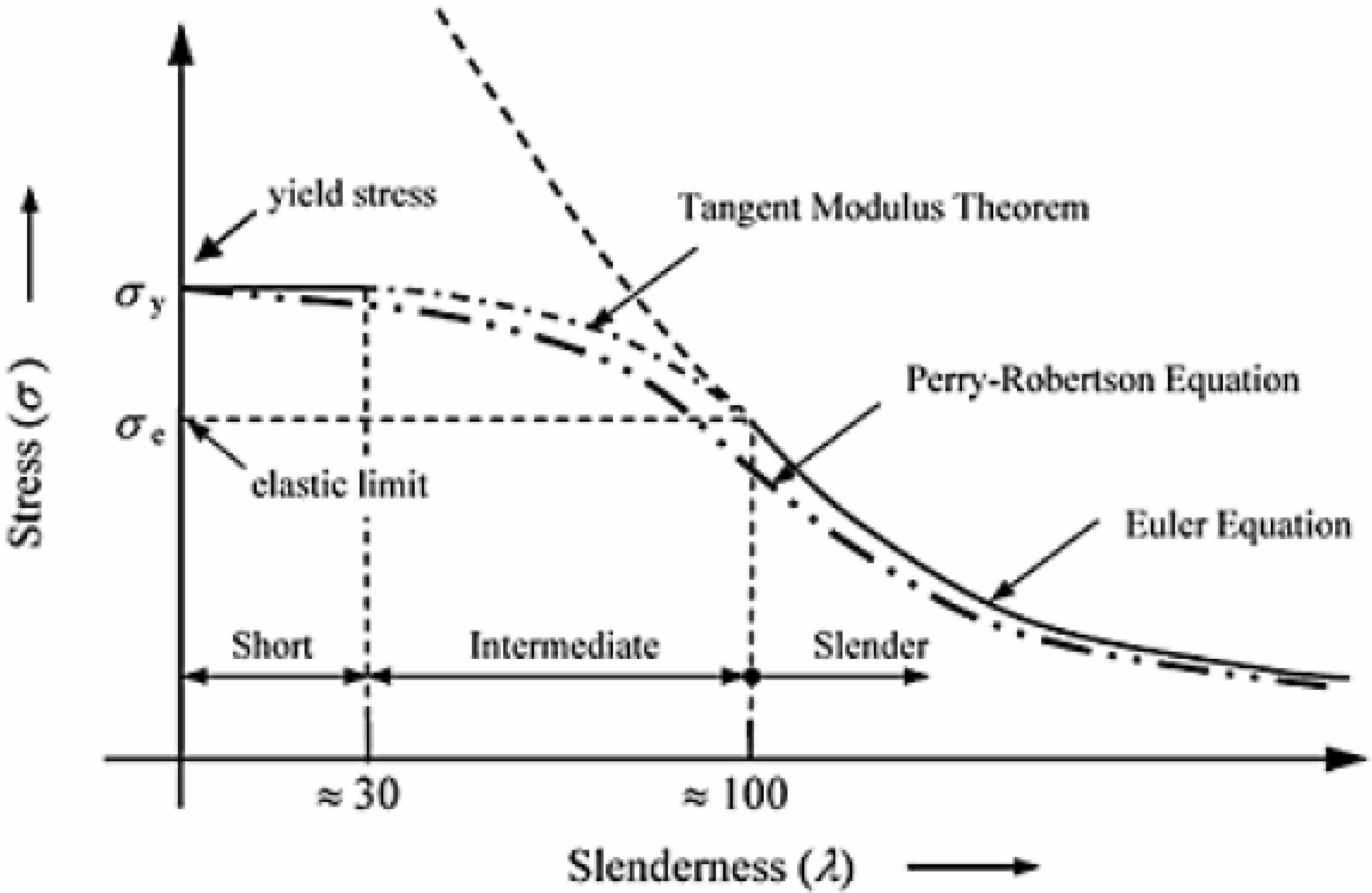
$$\sigma = \frac{1}{2}\left[\sigma_y + (1+\eta)\sigma_{cr}\right] - \sqrt{\frac{1}{4}\left[\sigma_y + (1+\eta)\sigma_{cr}\right]^2 - \sigma_y\sigma_{cr}} \qquad (48)$$

Experimental evidence obtained by Perry and Robertson indicated that for a mild steel the hypothetical initial curvature of the column could be represented as following.

$$\eta = 0.003\frac{L}{r} \qquad (49) \qquad\qquad \therefore \quad \sigma < \sigma_{cr} < \sigma_y$$

It is that value of slenderness ratio when the yield stress is first attained in one of the extreme fibres.

$$\sigma = \frac{1}{2}\left[\sigma_y + (1+0.003\frac{L}{r})\sigma_{cr}\right] - \sqrt{\frac{1}{4}\left[\sigma_y + (1+0.003\frac{L}{r})\sigma_{cr}\right]^2 - \sigma_y\sigma_{cr}} \qquad (50)$$



yield stress
Tangent Modulus Theorem
σ_y
Stress (σ)
σ_e
Perry-Robertson Equation
elastic limit
Euler Equation
Short
Intermediate
Slender
≈ 30
≈ 100
Slenderness (λ)