# Plain & Reinforced Concrete-1

## CE-314

Lecture # 9

# Flexural Analysis and Design of Beams

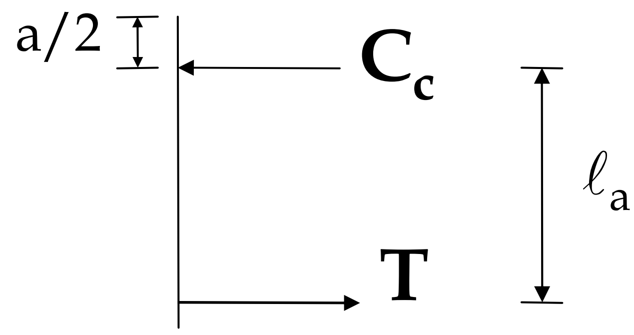
(Ultimate Strength Design of Beams)

# Plain & Reinforced Concrete-1

## Under-Reinforced Failure

### Stage-I, Un-cracked Section

N.A. position is fixed, means "$\ell_a$" remains constant. Only "T" and "$C_c$" increase with the increase of load




**Internal Force Diagram**

### Stage-II, Cracked Section

When section cracks, N.A. moves towards compression face means "$\ell_a$" increases. "T" and "$C_c$" also increase.

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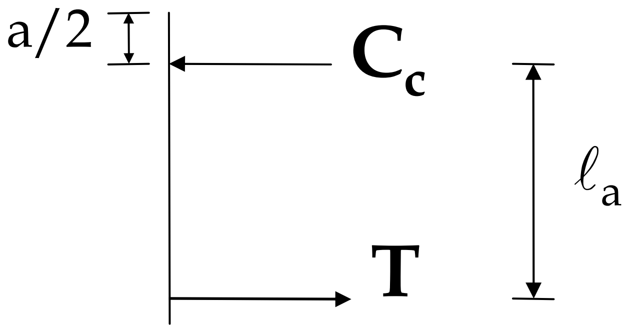
## Under-Reinforced Failure (contd…)

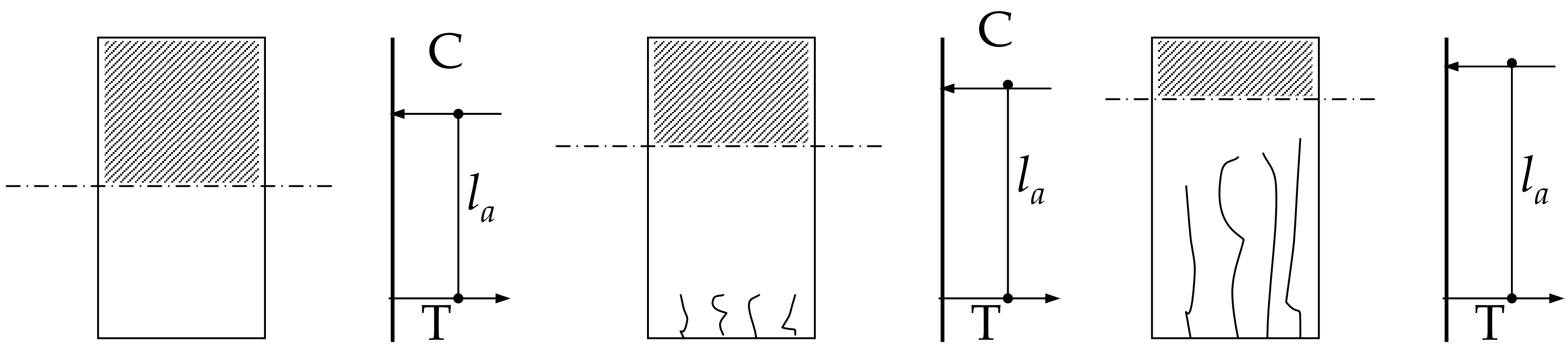
### Stage-III, Yielding in Steel Occur

$T = A_s f_y$ remains constant and $C_c$ also remains constant. "$\ell_a$" increases as the N.A. moves towards compression face because cracking continues.

**Failure initiates by the yielding of steel but final failure is still by crushing of concrete**




**Internal Force Diagram**




Stage 1: Uncracked Section

Stage 2: Cracked Section

Stage 3: Ultimate Condition

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## Under-Reinforced Failure (contd…)

### Derivation for ρ

Design Moment Capacity

$$\phi_b M_n = \phi_b T \times \ell_a$$

$$= \phi_b A_s f_y \times \left( d - \frac{a}{2} \right)$$

For tension controlled section $\phi = 0.9$

$$\phi_b M_n = 0.9 A_s f_y \times \left( d - \frac{a}{2} \right) \qquad \text{———— (1)}$$

And

$$a = \frac{A_s f_y}{0.85 f_c' b} \qquad \text{———— (2)}$$

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## Under-Reinforced Failure (contd…)

Put value of "a" from (1) to (2)

$$\phi_b M_n = 0.9 A_s f_y \left( d - \frac{A_s f_y}{2 \times 0.85 f_c' b} \right)$$

$$= 0.9 \times \rho b d \times f_y \left( d - \frac{\rho b d \times f_y}{2 \times 0.85 f_c' b} \right)$$

For economical design

$$\phi_b M_n = M_u$$

$$M_u = 0.9 \times \rho b d^2 \times f_y \left( 1 - \frac{\rho \times f_y}{2 \times 0.85 f_c'} \right)$$

$$\frac{M_u}{bd^2} = 0.9 \rho \times f_y \left( 1 - \frac{\rho}{2} \times \frac{f_y}{0.85 f_c'} \right)$$

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## Under-Reinforced Failure (contd…)

Let

$$\frac{M_u}{bd^2} = R \quad \text{(MPa)} \qquad \text{And} \qquad \frac{0.85fc'}{f_y} = \omega$$

Hence

$$R = 0.9\rho \times f_y\left(1 - \frac{\rho}{2\omega}\right)$$

$$\frac{R}{0.9f_y} = \rho\left(1 - \frac{\rho}{2\omega}\right)$$

$$\frac{R}{0.9f_y} = \rho - \frac{\rho^2}{2\omega}$$

$$\rho^2 - 2\omega \times \rho + \frac{2\omega \times R}{0.9f_y} = 0$$

$$\rho^2 - 2\omega \times \rho + \frac{2\omega \times R}{0.9f_y} \times \frac{0.85}{0.85} \times \frac{fc'}{fc'} = 0$$

$$\rho^2 - 2\omega \times \rho + \frac{\omega^2 \times R}{0.3825f_c'} = 0$$

$$\rho = \frac{2\omega \pm \sqrt{4\omega^2 - 4 \times \frac{R \times \omega^2}{0.3825fc'}}}{2}$$

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## Under-Reinforced Failure (contd…)

By simplification

$$\rho = \omega\left(1 \pm \sqrt{1 - \frac{R}{0.3825 f_c'}}\right)$$

We have to use –ve sign for under reinforced sections. So

$$\rho = \omega\left(1 - \sqrt{1 - \frac{2.614R}{f_c'}}\right)$$

Reason

For under reinforced section $\rho < \rho_b$

If we use positive sign $\rho$ will become greater than $\rho_b$, leading to brittle failure.

$$\rho_b = 0.85\frac{f_c'}{f_y}\;\beta_1 \times \frac{600}{600 + f_y}$$

$\omega$

$< 1.0$

$\rho_b < \omega$

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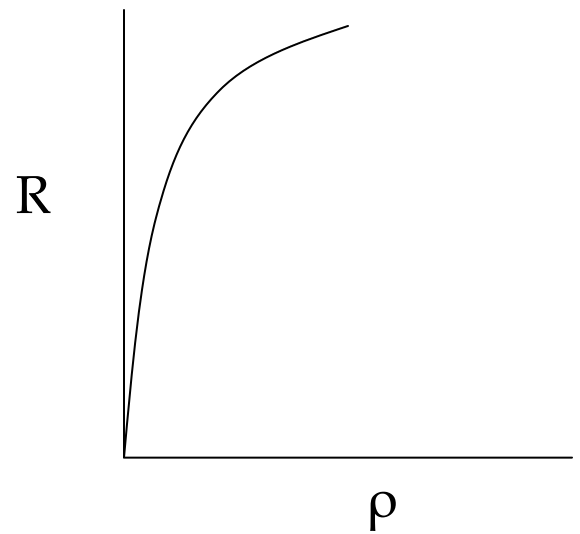
## Plotting of R -ρ

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{\rho b d}{\omega b} = \rho \times \frac{d}{\omega} \qquad \text{———} \qquad (3)$$

$$M_u = \phi_b \rho b d f_y \left( d - \frac{a}{2} \right) \quad \Rightarrow \quad \frac{M_u}{bd^2} = \phi_b \rho \times f_y \left( 1 - \frac{a}{2d} \right)$$

$$R = 0.9 \rho \times f_y \left( 1 - \frac{a}{2d} \right) \qquad \text{———} \qquad (4)$$

| | | | | |
|---|---|---|---|---|
| ρ | | | | |
| R | | | | |



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## Trial Method for the determination of "$A_s$"

$$a = \frac{A_s f_y}{0.85 f_c' b} \quad \text{———} \quad (A)$$

$$M_u = 0.9 A_s f_y \left( d - \frac{a}{2} \right) \quad \text{———} \quad (B)$$

$$A_s = \frac{M_u}{0.9 f_y \left( d - \frac{a}{2} \right)} \quad \text{———} \quad (C)$$

Trial # 1, Assume some value of "a" e.g. d/3 or d/4 or any other reasonable value, and put in (C) to get "$A_s$"

Trial # 2, Put the calculated value of "$A_s$" in (A) to get "a". Put this "a" value in (C) to get "$A_s$"

Keep on doing the trials unless "As" from a specific trial becomes equal to the "As" calculated from previous trial.

THIS VALUE OF $A_S$ WILL BE THE FINAL ANSWER.

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Is The Section Under-Reinforced or NOT ?

1. Calculate $\rho$ and if it is less than $\rho_{max}$, section is under reinforced
2. Using "a" and "d" calculate $\varepsilon_t$ if it is $\geq 0.005$, section is under-reinforced (tension controlled)
3. If section is over-reinforced than in the following equation –ve term will appear in the under-root.

$$\rho = \omega\left(1 - \sqrt{1 - \frac{2.614R}{f_c'}}\right)$$

# Plain & Reinforced Concrete-1

## Is The Section is Under-Reinforced or NOT ?

(contd…)

1. For tension controlled section, $\varepsilon_t = 0.005$, $a = \beta_1 \frac{3}{8} d$

Using formula of $M_n$ from concrete side

$$M_u = \phi_b M_n = \phi_b C_c \times \ell_a$$

$$M_u = 0.9 \times 0.85 f_c' ba \times \left( d - \frac{a}{2} \right)$$

$$M_u = 0.9 \times 0.85 f_c' b \left( 0.85 \frac{3}{8} d \right) \times \left( d - \frac{0.85 \frac{3}{8} d}{2} \right)$$

$$M_u = 0.205 f_c' bd^2$$

$$d_{min} = \sqrt{\frac{M_u}{0.205 f_c' \times b}}$$

If we keep $d > d_{min}$ the resulting section will be under-reinforced.

$d > d_{min}$ means that section is stronger in compression.

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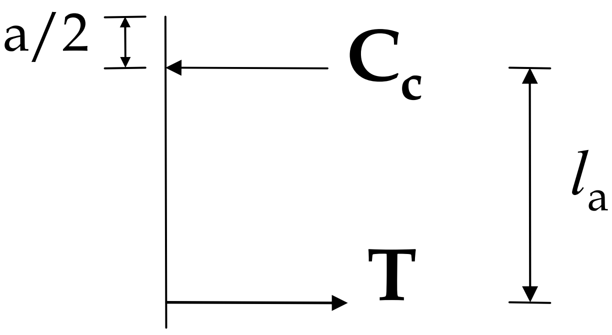
## Over-Reinforced Failure

### Stage-I, Un-cracked Section

### Stage-II, Cracked Section

These two stages are same as in under-reinforced section.

### Stage-III, Concrete reaches strain of 0.003 but steel not yielding




**Internal Force Diagram**

We never prefer to design a beam as over-reinforced (compression controlled) as it will show sudden failure.

$\phi = 0.65 \qquad \varepsilon_s < \varepsilon_y \quad f_s < f_y$

# Plain & Reinforced Concrete-1

## Over-Reinforced Failure

### Stage-III, Concrete reaches strain of 0.003 but steel not yielding (contd…)

$$\phi_b M_n = C_c \times \ell_a$$

$$\phi_b M_n = 0.65 \times 0.85 f_c' b a \times \left( d - \frac{a}{2} \right) \qquad \text{(i)}$$

"a" is unknown as "$f_s$" is not known

$$a = \frac{A_s f_s}{0.85 f_c' b} \qquad \text{(ii)}$$

# Plain & Reinforced Concrete-1

## Over-Reinforced Failure

Stage-III, Concrete reaches strain of 0.003 but steel not yielding (contd…)

Comparing ΔABC & ΔADE

$$\frac{\varepsilon_s}{0.003} = \frac{d-c}{c}$$

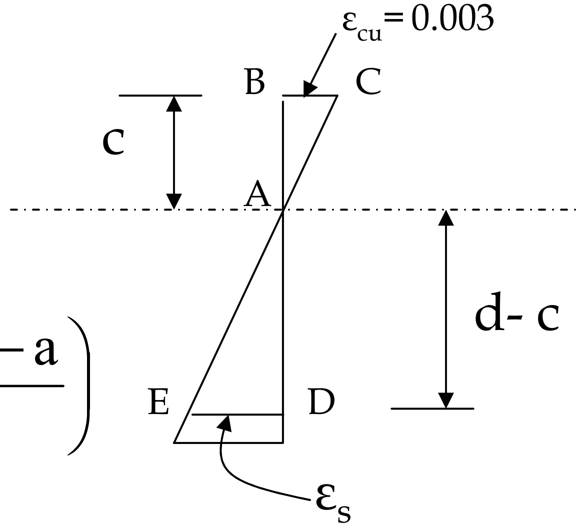
$$\frac{\varepsilon_s}{0.003} = \frac{d - a/\beta_1}{a/\beta_1}$$

$$\varepsilon_s = 0.003\left(\frac{\beta_1 d - a}{a}\right) \quad \text{(iii)}$$

$$f_s = E \times \varepsilon_s$$

$$f_s = 200{,}000 \times 0.003\left(\frac{\beta_1 d - a}{a}\right)$$

$$f_s = 600 \times \left(\frac{\beta_1 d - a}{a}\right) \quad \text{(iv)}$$




Strain Diagram

Eq # (iv) is applicable when $\varepsilon_s < \varepsilon_y$

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Putting value of "$f_s$" from (iv) to (ii)

$$a = \frac{A_s \times 600\left(\frac{\beta_1 d - a}{a}\right)}{0.85 f_c' b} \qquad \text{(v)}$$

Eq. # (v) is quadratic equation in term of "a".

**Flexural Capacity**

$$\phi_b M_n = \phi_b C_c\left(d - \frac{a}{2}\right) = \phi_b 0.85 f_c' ba\left(d - \frac{a}{2}\right)$$

$$\phi_b M_n = \phi_b\, T\left(d - \frac{a}{2}\right) = \phi_b A_s f_s\left(d - \frac{a}{2}\right)$$

**Calculate "a" from (v) and "$f_s$" from (iv) to calculate flexural capacity from these equations**

# Plain & Reinforced Concrete-1

| Extreme Tensile Steel Strain $\varepsilon_t$ | Type of X-section | c/d | a/d | $\rho_{max}$ | $\phi$ |
|---|---|---|---|---|---|
| $< \varepsilon_y$ | Compression Controlled | $> \left(\frac{600}{600+f_y}\right)$ | $> \beta_1\left(\frac{600}{600+f_y}\right)$ | $> \beta_1 \frac{0.85f_c'}{f_y}\left(\frac{600}{600+f_y}\right)$ | 0.65 |
| $\geq \varepsilon_y$ | Transition Section (Under-Reinforced) | $\leq \left(\frac{600}{600+f_y}\right)$ | $\leq \beta_1\left(\frac{600}{600+f_y}\right)$ | $\leq \beta_1 \frac{0.85f_c'}{f_y}\left(\frac{600}{600+f_y}\right)$ | 0.65 to 0.9 |
| $\geq 0.004$ | Under-Reinforced (minimum strain for beams) | $\leq \frac{3}{7}$ | $\leq \beta_1 \frac{3}{7}$ | $\leq \beta_1 \frac{0.85f_c'}{f_y} \times \frac{3}{7}$ | 0.65 to 0.9 |
| $\geq 0.005$ | Tension Controlled | $\leq \frac{3}{8}$ | $\leq \beta_1 \frac{3}{8}$ | $\leq \beta_1 \frac{0.85f_c'}{f_y} \times \frac{3}{8}$ | 0.9 |
| $\geq 0.0075$ | Redistribution is allowed | $\leq \frac{2}{7}$ | $\leq \beta_1 \frac{2}{7}$ | $\leq \beta_1 \frac{0.85f_c'}{f_y} \times \frac{2}{7}$ | 0.9 |

# Concluded